Dense graphs without C_7

1 Introduction

One of our main result in [2] reads as follows.

Theorem 1.1 ([2]). *For any $\varepsilon > 0$ and fixed integer $k \geqslant 3$, let G be an n-vertex maximal C_{2k-1}-free graph with minimum degree at least $(\frac{1}{2k-1} + \varepsilon)n$, then $G = H[\cdot]$ for some H, where $|H| \leqslant \mathrm{tw}_k(r)$, $r \leqslant e(d+1)(\frac{12ke}{\varepsilon})^d$ and $d \leqslant (2k-1)^3\binom{2k-2}{k-1}2^{(2k-1)^{2k-2}}$.*

For example, when $k = 4$, $|H| = 2^{2^{2^{2^{\mathrm{Poly}(1/\varepsilon)}}}}$. With a slightly modified proof compared to the one in our proof, we can further reduce this number k to $k-2$. However, this proof becomes increasingly cumbersome as k grows larger. In this note we present the proof for C_7-free graphs.

The following tools are useful.

Theorem 1.2 ([2]). *For integer $k \geqslant 2$ and any $\varepsilon > 0$, let G be an n-vertex maximal C_{2k-1}-free graph with minimum degree at least $(\frac{1}{2k-1} + \varepsilon)n$. Then $\mathrm{VC}(G) \leqslant (2k-1)^3\binom{2k-2}{k-1} \cdot 2^{(2k-1)^{2k-2}}$.*

Lemma 1.3 ([1]). *For fixed integer $k \geqslant 2$, let $\varepsilon > 0$ and G be an n-vertex C_{2k-1}-free graph with minimum degree $\delta(G) \geqslant (\frac{1}{2k-1} + \varepsilon)n$, then G is C_ℓ-free for every odd ℓ with $k \leqslant \ell \leqslant 2k-1$.*

Lemma 1.4 (Partition lemma [3]). *Let d be a positive integer and G be an n-vertex graph with VC-dimension at most d. Let $\mathcal{F} := \{N_G(v) : v \in V(G)\}$. Then for any $1 \leqslant a \leqslant n$, there is a partition $V(G) = V_1 \sqcup V_2 \sqcup \cdots \sqcup V_r$, where $r = e(d+1) \cdot (2e)^d \left(\frac{n}{a}\right)^d$, such that for each $i \in [r]$, any pair of vertices $u, v \in V_i$ satisfies that $|N_G(v) \triangle N_G(u)| \leqslant 2a$.*

2 Dense C_7-free graphs

Theorem 2.1. *For any $\varepsilon > 0$, let G be an n-vertex maximal C_7-free graph with minimum degree $\delta(G) \geqslant (\frac{1}{7} + \varepsilon)n$, then $G = H[\cdot]$ for some H, where $|H| \leqslant 2^{2^{(\frac{1000}{\varepsilon})^d}}$ and $d \leqslant 10^{10^6}$.*

Proof of Theorem 2.1. First by Lemma 1.3, G is also C_5-free. By Theorem 1.2, we can see $\mathrm{VC}(G) = d \leqslant 10^{10^6}$. Take $s = \frac{\varepsilon n}{100}$, by Lemma 1.4, $V(G)$ can be partitioned into $V_1 \sqcup \cdots \sqcup V_r$ with $r \leqslant 3e(\frac{2en}{s})^d \leqslant (\frac{999}{\varepsilon})^d$, such that any pair of vertices in the same part have at least $(\frac{1}{7} + \varepsilon)n - 2s$ common neighbors.

Claim 2.2. *$G[V_i]$ is an independent set for each $i \in [r]$.*

Proof of claim. Suppose $x, y \in V_i$ are adjacent in G, since they share at least $(\frac{1}{7} + \varepsilon)n - 2s > \frac{n}{7} > r + 2$ common neighbors, by pigeonhole principle, there are vertices $x', y' \in N(x) \cap N(y) \cap V_j$ for some $j \in [r]$, then one can find a vertex $z \in N(x') \cap N(y') \backslash \{x, y\}$ such that $xyy'zx'$ forms a copy of C_5, a contradiction to Lemma 1.3. ■

We then refine the above partition by the following rules. For each $i \in [r]$, we partition V_i into at most $m = 2^r$ parts, say $V_i := V_i^1 \sqcup \cdots \sqcup V_i^m$ such that for any $j \in [m]$ and $\ell \in [r]$, any pair of vertices $x, y \in V_i^j$ satisfy $N(x) \cap V_\ell = \varnothing$ if and only if $N(y) \cap V_\ell = \varnothing$. Furthermore, for each $i \in [r]$ and $j \in [m]$, we partition V_i^j into $q := 2^{rm}$ parts, say $V_i^j = V_i^{j,1} \sqcup \cdots \sqcup V_i^{j,q}$ such that for any $k \in [q]$ and $(a, b) \in [r] \times [m]$, any pair of vertices $x, y \in V_i^{j,k}$ satisfy $N(x) \cap V_a^b = \varnothing$ if and only if $N(y) \cap V_a^b = \varnothing$.

It suffices to show the following result.

Claim 2.3. *For each $i_1, i_2 \in [r]$, $j_1, j_2 \in [m]$ and $k_1, k_2 \in [q]$, $V_{i_1}^{j_1,k_1}$ and $V_{i_2}^{j_2,k_2}$ are either complete, or anti-complete.*

Proof of claim. Suppose there exist vertices $a \in V_{i_1}^{j_1,k_1}$ and $b_1, b_2 \in V_{i_2}^{j_2,k_2}$ such that $ab_1 \in E(G)$ and $ab_2 \notin E(G)$. Since G is a maximal C_7-free graph, there must exist a path $av_1v_2v_3v_4v_5b_2$ in G, where $v_p \in V_{a_p}^{b_p,c_p}$ for $p \in [5]$, with $a_p \in [r]$, $b_p \in [m]$, and $c_p \in [q]$. Then by pigeonhole principle, there is at least one pair of vertices $x, y \in \{a, v_1, v_2, v_3, v_4, v_5, b\}$ sharing at least $\frac{\varepsilon n}{\binom{7}{2}} = \frac{\varepsilon n}{21}$ common neighbors. We shall derive a contradiction in each case, before this, note by Claim 2.2, $i_1 \neq a_1$, $i_1 \neq i_2$, $i_2 \neq a_5$ and $a_s \neq a_{s+1}$ for each $s \in [4]$. Moreover, we establish the following simple observations.

Observation 2.4. *The followings hold.*

(1) $b_1v_2, b_1v_3, b_2v_1, b_2v_2, b_2v_3 \notin E(G)$.

(2) *If $v_1 \in V_{a_5}$, then $b_1v_1 \notin E(G)$.*

(3) *If $v_4 \in V_{i_i}$, then $b_2v_4 \notin E(G)$.*

Proof of Observation 2.4. For (1): If $b_1v_2 inE(G)$, then one can pick a vertex $b' \in N(b_1) \cap N(b_2)$ so that $b'b_1v_2v_3v_4v_5b_2$ forms a copy of C_7, a contradiction. If $b_1v_3 \in E(G)$, then $b_1av_1v_2v_3$ forms a copy of C_5, a contradiction. If $b_2v_1 \in E(G)$, then one can pick a vertex $b' \in N(b_1) \cap N(b_2)$ so that $b_1av_1b_2b'$ forms a copy of C_5, a contradiction. If $b_2v_2 \in E(G)$, then $b_2v_2v_3v_4v_5$ forms a copy of C_5, a contradiction. If $b_2v_3 \in E(G)$, then one can pick a vertex $b' \in N(b_1) \cap N(b_2)$ so that $b_1av_1v_2v_3b_2b'$ forms a copy of C_7, a contradiction.

For (2), if $v_1 \in V_{a_5}$ and $b_1v_1 \in E(G)$, then $|N(v_1) \cap N(v_5)| \geqslant (\frac{1}{7} + \varepsilon)n - 2s$ and $|N(b_1) \cap N(b_2)| \geqslant (\frac{1}{7} + \varepsilon)n - 2s$, which together yield that we can pick a vertex $w \in N(v_1) \cap N(v_5)$ and a vertex $b' \in N(b_1) \cap N(b_2)$ so that $b_1av_1wv_5b_2b'$ forms a copy of C_7, a contradiction.

For (3), if $v_4 \in V_{i_1}$ and $b_2v_4 \in E(G)$, then we can find a vertex $w \in N(a) \cap N(v_4)$ and a vertex $b' \in N(b_1) \cap N(b_2)$ so that $awv_4v_5b_2b'b_1$ forms a copy of C_7, a contradiction. ■

- **Type 1:** If the pair (x, y) belongs to $\{(a, v_1), (v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_5), (v_5, b_2)\}$, then similar as the argument in Claim 2.2, there exists a pair of vertices in $N(x) \cap N(y)$ sharing at least $(\frac{1}{7} + \varepsilon)n - 2s > r + 2$ many common neighbors, which yields the existence of C_5 in G, a contradiction. In particular, when $(x, y) = (a, b_2)$, then since $b_1, b_2 \in V_{i_2}^{j_2,k_2}$, we can see $|N(a) \cap N(b_1)| \geqslant \frac{\varepsilon n}{21} - 2s \geqslant \frac{\varepsilon n}{50}$, then the similar argument derives the existence of C_5, a contradiction.

- **Type 2:** If the pair belongs to $\{(a, v_5), (b_2, v_4), (v_3, v_5), (v_2, v_4), (v_1, v_3), (a, v_2), (b_2, v_1)\}$,
  - (a, v_5): One can find a vertex $x_1 \in N(a) \cap N(v_5)$ such that $av_1v_2v_3v_4v_5x_1$ forms a copy of C_7, a contradiction.
  - (b_2, v_4): Since $b_1, b_2 \in V_{i_2}^{j_2,k_2}$, we have $|N(b_1) \cap N(b_2) \cap N(v_4)| \geqslant \frac{\varepsilon n}{21} - 2s \geqslant \frac{\varepsilon n}{50}$. Then there exists a vertex $x_2 \in N(b_1) \cap N(v_4)$ so that $av_1v_2v_3v_4x_2b_1$ forms a copy of C_7, a contradiction.

- (v_3, v_5): Note that b_1 has a neighbor $x_3 \in V_{a_5}^{b_5}$, if $x_3 = v_5$, then $b_1 a v_1 v_2 v_3 v_4 v_5$ forms a copy of C_7, a contradiction. Then by Observation 2.4, $x_3 \notin \{v_1, v_2, v_3, v_4, v_5\}$. Since $|N(x_3) \cap N(v_3) \cap N(v_5)| \geqslant \frac{\varepsilon n}{21} - 2s > 0$, we can pick a vertex $y_1 \in N(x_3) \cap N(v_3)$ so that $a v_1 v_2 v_3 y_1 x_3 b_1$ forms a copy of C_7, a contradiction.
- (v_2, v_4): Note that b_1 has a neighbor $x_4 \in V_{a_5}^{b_5}$, if $x_4 = v_5$, then $b_1 a v_1 v_2 v_3 v_4 v_5$ forms a copy of C_7, a contradiction. Then by Observation 2.4, $x_4 \notin \{v_1, v_2, v_3, v_4, v_5\}$. Moreover, we can also find a neighbor of x_4, say $y_2 \in N(x_4) \cap V_{a_4}$, if $y_2 = v_4$, then $b_1 a v_1 v_2 v_3 v_4 x_4$ forms a copy of C_7, a contradiction. If $y_2 = v_2$, then $x_4 b_1 a v_1 v_2$ forms a copy of C_5, a contradiction. Therefore $y_2 \notin \{v_2, v_4\}$, then since $|N(y_2) \cap N(v_2) \cap N(y_4)| \geqslant \frac{\varepsilon n}{21} - 2s$, we can can pick a vertex $z_1 \in N(y_2) \cap N(v_2)$ so that $a v_1 v_2 z_1 y_2 x_4 b_1$ forms a copy of C_7, a contradiction.
- (v_1, v_3): Note that b_2 has a neighbor $x_5 \in V_{i_1}^{j_1}$, if $x_5 = v_5$, then we can find a vertex $w \in N(a) \cap N(v_5)$ so that $a v_1 v_2 v_3 v_4 v_5 w$ forms a copy of C_7, a contradiction. Therefore, by Observation 2.4 we have $x_5 \notin \{v_1, v_2, v_3, v_4, v_5\}$. Also note that x_5 has a neighbor $y_3 \in V_{a_1}$, if $y_3 = v_1$, then $x_5 v_1 v_2 v_3 v_4 v_5 b_2$ forms a copy of C_7. If $y_3 = v_3$, then $b_2 x_5 v_3 v_4 v_5$ forms a copy of C_5, a contradiction. Therefore we can see $|N(y_3) \cap N(v_3)| \geqslant \frac{\varepsilon n}{21} - 2s$ which yields that we can pick a vertex $z_2 \in N(y_3) \cap N(v_3)$ so that $b_2 x_5 y_3 z_2 v_3 v_4 v_5$ forms a copy of C_7, a contradiction.
- (a, v_2): Note that b_2 has a neighbor $x_6 \in V_{i_1}^{j_1}$, if $x_6 = v_5$ then $|N(v_2) \cap N(v_5)| \geqslant \frac{\varepsilon n}{21} - 2s \geqslant \frac{\varepsilon n}{50}$, then one can find a vertex $w \in N(v_2) \cap N(v_5)$ so that $w v_2 v_3 v_4 v_5$ forms a copy of C_5, a contradiction. Therefore by Observation 2.4, $x_6 \notin \{v_1, v_2, v_3, v_4, v_5\}$. Then since $|N(x_6) \cap N(v_2)| \geqslant \frac{\varepsilon n}{21} - 2s \geqslant \frac{\varepsilon n}{50}$, there is a vertex $y_4 \in N(x_6) \cap N(v_2)$ so that $x_6 y_4 v_2 v_3 v_4 v_5 b_2$ forms a copy of C_7, a contradiction.
- (b_2, v_1): One can find a vertex $x_7 \in N(b_2) \cap N(v_1)$ so that $x_7 v_1 v_2 v_3 v_4 v_5 b_2$ forms a copy of C_7, a contradiction.

- **Type** 3**:**
  - If one of $\{(b_2, v_3), (v_2, v_5), (v_1, v_4), (a, v_3), (b_2, v_2)\}$ occurs, then one can easily pick a vertex in the common neighbor and obtain a copy of C_5, a contradiction.
  - (a, v_4): One can pick a vertex $x_8 \in N(a) \cap N(v_4)$ and a vertex $b' \in N(b_1) \cap N(b_2)$ so that $b_1 a x_8 v_4 v_5 b_2 b'$ forms a copy of C_7, a contradiction.
  - (v_1, v_5): One can pick a vertex $x_9 \in N(v_1) \cap N(v_5)$ and a vertex $b' \in N(b_1) \cap N(b_2)$ so that $b_1 a v_1 x_9 v_5 b_2 b'$ forms a copy of C_7, a contradiction.

This completes the proof. ■

By Claim 2.2 and the rules of partition, we can see G is a blowup of H such that $|H| \leqslant rmq = r \cdot 2^r \cdot 2^{r2^r} \leqslant 2^{2^{(\frac{1000}{\varepsilon})^d}}$, where $d \leqslant 10^{10^6}$ by Theorem 1.2. □

References

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