Sharp results for comparability graphs avoiding Dilworth Theorem

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1 Alternative proof

Theorem 1.1. *Let G be an n-vertex comparability graph, then $h(G) \leqslant \frac{n}{\alpha(G)}$. In particular, if $\alpha(G) = cn$ for some $c > 0$, then $h(G) \leqslant \lfloor \frac{1}{c} \rfloor$.*

Proof of Theorem 1.1. Let P be a poset and G be the comparability graph of P with $\alpha(G) = \alpha$. If the number of maximum independent sets at most $\frac{n}{\alpha}$, the problem is trivial. Therefore we can always assume the number of maximum independent sets is larger than $\frac{n}{\alpha}$. For any pair of distinct $I_1, I_2 \in \mathcal{I}_{\max}(G)$, we say that I_1 *supersedes* I_2, denoted by $I_1 \succeq I_2$, if for each vertex v in $I_1 \backslash I_2$, there exists some vertex v' in $I_2 \backslash I_1$ such that $v \succ v'$. If I_1 neither supersedes I_2 nor is superseded by I_2, we say I_1 and I_2 are *unrelated*.

Our proof consists of two parts, the first part is to establish a series of useful structural properties, while in the second part, we provide a switch algorithm and show the existence of small hitting set.

The first structural property focus on the induced subgraph $G[I_1 \triangle I_2]$ when $I_1 \succeq I_2$.

Lemma 1.2. *Let G be a comparability graph and let $I_1, I_2 \in \mathcal{I}_{\max}(G)$ such that $I_1 \succeq I_2$, then there does not exist any pair of vertices $(v \neq v')$ with $v \in I_1$ and $v' \in I_2$ such that $v \neq v'$ and $v \prec v'$. Moreover, there is a perfect matching M between $I_1 \backslash I_2$ and $I_2 \backslash I_1$, where $M = \{a_i b_i\}_{1 \leqslant i \leqslant |I_1 \backslash I_2|}$ and $a_i \succ b_i$ for any $1 \leqslant i \leqslant |I_1 \backslash I_2|$.*

Proof of Lemma 1.2. Suppose that there areis a pair $(v \neq v')$ with $v \in I_1$, and $v' \in I_2$ such that $v \neq v'$ and $v \prec v'$. Then we must have $v \in I_1 \backslash I_2$ and, since $I_1 \succeq I_2$, there exists another vertex v'' such that $v \succ v''$, which implies $v' \succ v''$ in I_2. This contradicts that I_2 is an independent set.

Let S be a subset of $I_1 \backslash I_2$, we define $N_{I_2 \backslash I_1}(S) := \{u \in I_2 \backslash I_1 : \exists\, v \in I_1 \backslash I_2 \text{ such that } uv \in E(G)\}$. Note that for any $S \subseteq I_1 \backslash I_2$, $|N_{I_2 \backslash I_1}(S)| \geqslant |S|$, for otherwise $(I_2 \cup S) \backslash N_{I_2 \backslash I_1}(S)$ is an independent set of size larger than α, a contradiction. Then by Hall's theorem [?], there exists a perfect matching $M = \{a_i b_i\}_{i \in [|I_1 \backslash I_2|]}$ between $I_1 \backslash I_2$ and $I_2 \backslash I_1$. By the first part, we see that $a_i \succ b_i$ as desired. □

Next, we would like to demonstrate that the supersedence relation $\succeq$ is transitive.

Proposition 1.3. *Let $I_1, I_2, I_3 \in \mathcal{I}_{\max}(G)$, the followings hold.*

(1) *If $I_1 \succeq I_2$ and $I_2 \succeq I_3$, then $I_1 \succeq I_3$.*

(2) *If $I_1 \succeq I_2, I_3$, then for any $I \in \mathcal{I}_{\max}(G)$ such that $I \subseteq I_2 \cup I_3$, we have $I_1 \succeq I$.*

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Proof of Proposition 1.3. For (1), let $I_1, I_2, I_3 \in \mathcal{I}_{\max}(G)$ such that $I_1 \succeq I_2$, and $I_2 \succeq I_3$. We select an arbitrary vertex v in $I_1 \backslash I_3$, if v belongs to $I_2 \backslash I_3$, then since $I_2 \succeq I_3$, there is a vertex v' in I_3 such that $v \succ v'$, we are done. Otherwise, v does not belong to I_2, then as $I_1 \succeq I_2$, there is a vertex v'' in I_2 such that $v \succ v''$. Similarly since $I_2 \succeq I_3$, if v'' is in I_3 then we are done, otherwise, there is a vertex $v''' \in I_3$ such that $v'' \succ v'''$, which implies that $v \succ v'''$. Therefore, $I_1 \succeq I_3$.

For (2), suppose that $I_1 \succeq I_2, I_3$ and $I \in \mathcal{I}_{\max}(G)$ such that $I \subseteq I_2 \cup I_3$. Then for any vertex $v \in I_1 \backslash I$, v has at least one neighborhood v' in I since I and I_1 are both maximum independent sets. Then by Lemma 1.2, Since $I \subseteq I_2 \cup I_3$, and I_1 supersedes both of I_2 and I_3, by Lemma 1.2, there is no vertex $u \in I \cap N_G(v)$ such that $v \prec u$. Therefore, $v \succ v'$ and so $I_1 \succeq I$must hold. This finishes the proof. □

Building upon the preparatory work we have done so far, we are now poised to present a structural result concerning comparability graphs, which is highly beneficial for the proof. Let $I_1, I_2 \in \mathcal{I}_{\max}(G)$, we define the subsets $I_1^+, I_1^- \subseteq I_1$ and $I_2^+, I_2^- \subseteq I_2$ as follows.

$$I_1^+ := \{v \in I_1 : \text{there exists some vertex } v' \in I_2 \text{ such that } v \succ v'\};$$

$$I_1^- := \{v \in I_1 : \text{there exists some vertex } v' \in I_2 \text{ such that } v' \succ v\};$$

$$I_2^+ := \{u \in I_2 : \text{there exists some vertex } u' \in I_1 \text{ such that } u \succ u'\};$$

$$I_2^- := \{u \in I_2 : \text{there exists some vertex } u' \in I_1 \text{ such that } u' \succ u\}.$$

Lemma 1.4. *Let $I_1, I_2 \in \mathcal{I}_{\max}(G)$ such that I_1 and I_2 are unrelated. Then I_1 can be partitioned into three parts $I_1 \cap I_2$, I_1^+ and I_1^- while I_2 can also be partitioned into three parts $I_1 \cap I_2$, I_2^+ and I_2^- such that the following properties hold.*

(1) *There is no edge between I_1^+ and I_2^+, also, there is no edge between I_1^- and I_2^-.*

(2) *For any edge $vv' \in E(G)$ such that $v \in I_1^+$ and $v' \in I_2^-$, we have $v \succ v'$. Similarly, for every edge $uu' \in E(G)$ such that $u \in I_2^+$ and $u' \in I_1^-$, we have $u \succ u'$.*

(3) *There is a perfect matching between I_1^+ and I_2^-, and a perfect matching between I_1^- and I_2^+. Moreover, we have $|I_1^+| = |I_2^-|$, $|I_1^-| = |I_2^+|$.*

Proof of Lemma 1.4. First, we can see $I_1^+ \cap I_1^- = \emptyset$, otherwise $v \in I_1^+ \cap I_1^-$ yields there are vertices $v', v'' \in I_2$ such that $v' \prec v \prec v''$, which contradicts I_2 is an independent set. Similarly $I_2^+ \cap I_2^- = \emptyset$, therefore I_1 can be partitioned into $I_1^+ \cup I_1^- \cup (I_1 \cap I_2)$ and I_2 can be partitioned into $I_2^+ \cup I_2^- \cup (I_1 \cap I_2)$.

For (1), suppose there is an edge $vu \in E(G)$ with $v \in I_1^+$ and $u \in I_2^+$. Then there is a vertex $v' \in I_2$ such that $v \succ v'$, and there is a vertex $u' \in I_1$ such that $u \succ u'$. One can easily check that exactly one of the events $v \succ u \succ u'$ and $u \succ v \succ v'$ occurs, which implies either $vu' \in E(G)$ or $uv' \in E(G)$, a contradiction to that both of I_1 and I_2 are independent sets.

For (2), suppose there is an edge $vv' \in E(G)$ such that $v \in I_1^+$, $v' \in I_2^-$ and $v \prec v'$, then by definition of I_2^-, there exists some vertex $v'' \in I_1$ such that $v' \prec v''$, which also yields $v \prec v''$, a contradiction to that I_1 is an independent set. The other statement in (2) holds for the same reason, we omit the repeated argument.

For (3), suppose that $|I_1^+| > |I_2^-|$, then based on (1), $I_1^+ \cup I_2^+ \cup (I_1 \cap I_2)$ is also an independent set. Moreover, $|I_1^+ \cup I_2^+ \cup (I_1 \cap I_2)| > |I_2|$, which is a contradiction to $I_2 \in \mathcal{I}_{\max}(G)$. Therefore, we have $|I_1^+| = |I_2^-|$, $|I_1^-| = |I_2^+|$. By Lemma 1.2, there is a perfect matching between $I_1 \backslash I_2$ and $I_2 \backslash I_1$. By (1), we can further see that there is a perfect matching between I_1^+ and I_2^- and a perfect matching between I_1^- and I_2^+.

□

With the above structural properties in hand, we then show there exists a hitting set of size at most $\frac{n}{\alpha(G)}$. Despite this part being quite short, it might help to briefly outline the main ideas. Our strategy roughly consists of three parts, the first is to iteratively find a sequence of maximum independent sets $I_1, I_2, \ldots, I_k$ of maximal length such that for any $1 \leqslant i \leqslant k$, I_i supersedes all I_j with $i < j$. The second step is to show, that for any other $I \in \mathcal{I}_{\max}(G)$, I should be a subset of $\bigcup_{i=1}^{k} I_i$, which guarantees that all of the vertices belonging to some maximum independent set also belong to $\bigcup_{i=1}^{k} I_i$. Finally, we will build $\frac{n}{\alpha(G)}$ boxes so that the vertices in any maximum independent set must belong to those boxes, which provides the desired upper bound $\frac{n}{\alpha(G)}$ by pigeonhole principle.

To select a sequence of maximum independent sets of maximal length such that for any $1 \leqslant i \leqslant k$, $I_i \succeq I_j$ for all $i < j$, we apply the following switch algorithm. First, we take an arbitrary maximum independent set $I_0 \in \mathcal{I}_{\max}(G)$, then consider another arbitrary maximum independent set I, if $I \succeq I_0$ or $I_0 \succeq I$, we just put I and I_0 into our selected sequence, and denote them as A_1 and A_2, where $A_1 \succeq A_2$. Otherwise, if I and I_0 are unrelated, we partition both of them, namely $I = I^- \cup I^+ \cup (I_0 \cap I)$ and $I_0 = I_0^- \cup I_0^+ \cup (I \cap I_0)$ according to Lemma 1.4. Now we can find a pair of different maximum independent sets I' and I_0' in G, where $I' = (I \cup I_0^-)\backslash I^+$ and $I_0' = (I_0 \cup I^+)\backslash I_0^-$. By definitions of I_0^- and I^+, we can see that $I_0' \succeq I'$. Moreover, we have $I \cup I_0 \subseteq I_0' \cup I'$, and actually here $I \cup I_0 = I_0' \cup I'$. Then we put I_0' and I' into our selected sequence and denote $A_1 = I_0'$ and $A_2 = I'$.

More generally, for $t \geqslant 2$, assume we have already selected a sequence of maximum independent sets $A_1, A_2, \ldots, A_t$ such that $A_i \succeq A_j$ for any $1 \leqslant i < j \leqslant t$. Then let A be a maximum independent set which is different from any A_i and also is not considered during the process of selecting A_i, for $1 \leqslant i \leqslant t$. If there exists some index $0 \leqslant j \leqslant t$ such that $A \succeq A_{j+1}$ and $A_j \succeq A$, (in particular, $j = 0$ means $A \succeq A_1$, and $j = t$ means A_t supersedes A) then we obtain a new sequence namely $A_1, \ldots, A_i, A, A_{j+1}, \ldots, A_t$. Otherwise, there exists some integer $0 \leqslant m \leqslant t$ such that A and A_m are unrelated, let m be the smallest index such that A and A_m are unrelated. We then produce the same switch operation as the previous, that is, we partition both of $A = A^- \cup A^+ \cup (A_m \cap A)$ and $A_m = A_m^- \cup A_m^+ \cup (A \cap A_m)$ according to Lemma 1.4 and then find a pair of new maximum independent sets A' and A_m', where $A' = (A \cup A_m^-)\backslash A^+$ and $A_m' = (A_m \cup A^+)\backslash A_m^-$.

For the original sequence of maximum independent sets $A_1, A_2, \ldots, A_t$ and A_m' and A', we have the following properties.

Claim 1.5. *Let $A' = (A \cup A_m^-)\backslash A^+$ and $A_m' = (A_m \cup A^+)\backslash A_m^-$ be two new maximum independent sets, then the followings hold.*

(1) *$A_m' \cup A' = A_m \cup A$, and in particular, $A_m' \cup A'$ is a subset of $\bigcup_{i=1}^{t} A_i$.*

(2) *$A_m' \succeq A'$;*

(3) *For any $1 \leqslant i < m$, $A_i \succeq A'$ and A_m'.*

(4) *For any integer $j > m$, $A_m' \succeq A_j$.*

Proof of claim. (1) and (2) are simple consequences from the definitions.

For (3), note that $i < m$ yields $A_i \succeq A_m$ by the rules we select the sequence $A_1, \ldots, A_t$, moreover, as m is the smallest index such that A and A_m are unrelated, we also have $A_i \succeq A$ for any $i < m$. Since $A', A_m' \subseteq A \cup A_m$, then by Proposition 1.3(2), A_i supersedes both of A' and A_m'.

For (4), As $A_m' = (A_m \cup A^+)\backslash A_m^-$ and there is a perfect matching between A^+ and A_m^-, we can easily see that $A_m' \succeq A_m$ by definition. Then (4) holds by Proposition 1.3(1). ■

Then we can replace A_m' with A_m, obtaining a new sequence of maximum independent sets $A_1, A_2, \ldots, A_{m-1}, A_m', A_{m+1}, \ldots, A_t$, where $A_{m-1} \succeq A_m'$ and $A_m' \succeq A_{m+1}$. By Claim 1.5(2), we already know that $A'_m \succeq A'$. Therefore, next we need to apply the same operation on A' depending on whether there exists some $m+1 \leqslant j \leqslant t$ such that A' and A_j are unrelated. To better understand this algorithmic step, we provide a simple graphical depiction of the process, see Fig. 1.1.

We will execute the algorithm using all independent sets in the graph G. Through our analysis, we will ultimately identify a sequence of maximum independent sets $I_1, I_2, \ldots, I_k$, satisfying the condition that $I_i \succeq I_j$ for any positive integers $1 \leqslant i < j \leqslant k$. Furthermore, by Claim 1.5, we can see for any $I \in \mathcal{I}_{\max}(G)$, it either equals I_s for some $1 \leqslant s \leqslant k$, or I is a subset of the union of vertex sets $\bigcup_{i=1}^{k} I_i$.

It is now time to construct the hitting set. Initially, we assign distinct colors to each vertex v_i in $I_1 := v_1, v_2, \ldots, v_\alpha$ using color $i \in [\alpha]$. Leveraging Lemma 1.2, we establish that for any $1 \leqslant s \leqslant k-1$, there exists a perfect matching $M_{s,s+1}$ between $I_s \backslash I_{s+1}$ and $I_{s+1} \backslash I_s$. Let these perfect matchings be denoted as $M_{1,2}, M_{2,3}, \ldots, M_{k-1,k}$. Next, we proceed to color the vertices $u \in I_2 \backslash I_1$ with color j if uv_j forms an edge in the selected perfect matching $M_{1,2}$. For larger $s \geqslant 2$, following analogous rules, we subsequently color the vertices v in $I_{s+1} \backslash (\bigcup\limits_{i=1}^{s} I_i)$ using the the same color as w, if vw is an edge in the perfect matching $M_{s,s+1}$ and $w \in I_s \backslash I_{s+1}$.

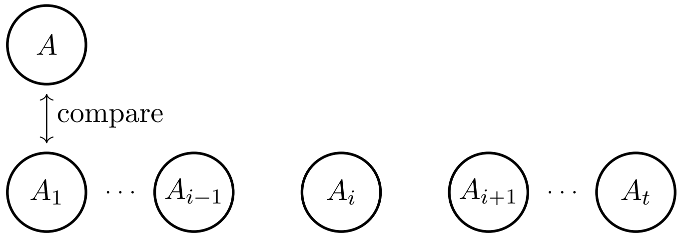
The following observation holds importance.

Claim 1.6. *The vertices receiving the same color $i \in [\alpha]$ form a clique.*

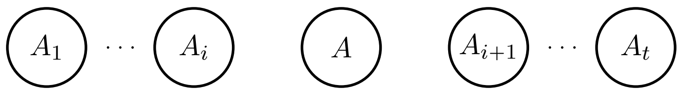
Proof of claim. As $I_i \succeq I_j$ holds for any distinct $1 \leqslant i < j \leqslant k$, we can see that in each perfect matching $M_{s,s+1}$ with vertices $a_1, a_2, \ldots, a_h \in I_s \backslash I_{s+1}$, $b_1, b_2, \ldots, b_h \in I_{s+1} \backslash I_s$ and edges $a_q b_q \in E(G)$, we have $a_q \succ b_q$ holds for any $1 \leqslant q \leqslant h$ by Lemma 1.2. That means, for each class $i \in [\alpha]$ with the vertices $c_{j_1}, \ldots, c_{j_\ell}$, where $j_1 < j_2 < \cdots < j_\ell$, $\ell \leqslant k$ and $c_{j_r} \in I_{j_r}$ for $1 \leqslant r \leqslant \ell$, we have $c_{j_1} \succ c_{j_2} \succ \cdots \succ c_{j_\ell}$. Therefore, each color class forms a copy of clique, which finishes the proof. ■

Now, we establish that there are a total of α color classes, each forming a clique. Furthermore, for each $I \in \mathcal{I}_{\max}(G)$, by Claim 1.5, I is a subset of $\bigcup_{i=1}^{k} I_i$. Notice that I can intersect each color class by at most one vertex, since I is an independent set and each color class forms a clique by Claim 1.6. Moreover, given that $|I| = \alpha$ and each vertex in I receives a color, we can see that I intersects each color class by precisely one vertex. Consequently, the vertices in the same color class hit all maximum independent sets. By the pigeonhole principle, the desired result $h(G) \leqslant \frac{n}{\alpha}$ follows.

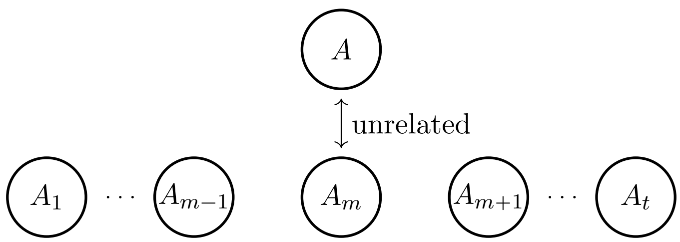
□



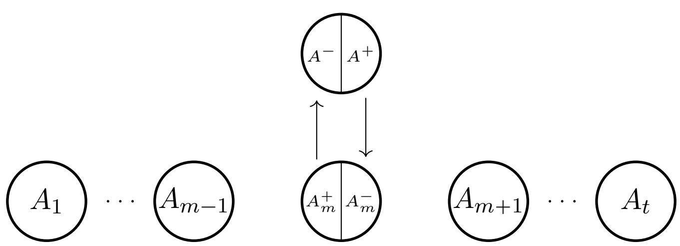
(a) Compare A with all selected maximum independent sets, if there exists some $1 \leqslant i \leqslant t$ such that $A_i \succeq A$ and $A \succeq A_{i+1}$, then turn to (b), otherwise turn to (c).



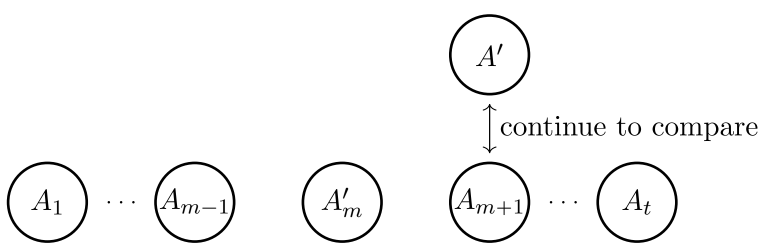
(b) If there exists some $1 \leqslant i \leqslant t$ such that $A_i \succeq A$ and $A \succeq A_{i+1}$, then we put A into the sequence and obtain a new sequence of $t+1$ many maximum independent sets.



(c) Suppose m is the smallest index such that A and A_m are unrelated, then turn to the next step (d).



(d) By Lemma 1.4, we can partition A into $A^+ \cup A^- \cup (A \cap A_m)$ and partition A_m into $A_m^+ \cup A_m^- \cup (A \cap A_m)$, and switch to obtain two new maximum independent sets A' and A'_m.



(e) Replace $A'_m := (A_m \cup A^+) \backslash A_m^-$ with A_m to obtain a new sequence of maximum independent sets, and then continue to check whether $A' := (A \cup A_m^-) \backslash A^+$ and A_j are unrelated for $j > m$ (turn to (a)).

Figure 1.1: A simple illustration of the switch algorithm