An alternative proof via regularity lemma

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1 Second proof via regularity lemma

Theorem 1.1. *Given $r \geqslant 3, \varepsilon > 0$ and G be an n-vertex K_r-free graph. If for every non-adjacent pair of vertices $u, v \in V(G)$, the induced subgraph $G[N(u) \cap N(v)]$ contains εn^{r-2} copies of K_{r-2}, then $G = F[\cdot]$ for some maximal K_r-free graph F on at most $2^{(r+\frac{1}{\varepsilon})^{2+o(1)}}$ vertices.*

1.1 Overview of the proof

This proof roughly consists of three parts.

1. As the graph G has bounded VC-dimension, Theorem 1.4 yields $V(G)$ can be equitably partitioned into K parts $V_1, \ldots, V_K$ such that almost all pairs are λ-homogeneous. We further prove that, after moving a small proportion of vertices to an exceptional set V_0, we obtain a refined partition $V(G)\backslash V_0 = U_1 \cup \cdots \cup U_M$ with $M \leqslant K$ such that (U_i, U_j) can be either very dense or very sparse.

2. We continue to refine the partition, more precisely, for those non-adjacent pair of vertices which belong to dense pairs, we move both of them to the exceptional set V_0. We can prove that the number of these movements is small, that is, $|V_0|$ is small. After moving all vertices in those parts with relatively small cardinality, we can prove that the remaining parts $Z_1, \ldots, Z_J$ in $V(G)\backslash V_0$ with $J \leqslant M \leqslant K$ are pairwise complete or anti-complete.

3. The final task is to deal with the vertices in V_0, we can show that V_0 can be partitioned in to at most $T \leqslant 2^J \leqslant 2^K$ parts $H_1, \ldots, H_T$ such that $Z_1, \ldots, Z_M, H_1, \ldots, H_T$ are pairwise complete or anti-complete, which implies that G can be viewed as a blow-up of K_r-free graph with at most $J + T \leqslant K + 2^K$ vertices. More carefully, we can improve the upper bound $T \leqslant K^{\text{poly}(1/\varepsilon, r)}$ by the so-called Sauer-Shelah lemma.

We will take advantage of the following results in our proof.

Proposition 1.2. *Let G be a graph and $V_1, \ldots, V_r$ be pairwise disjoint subsets of $V(G)$, each of size m. Suppose that for every $1 \leqslant i < j \leqslant r$, $d(V_i, V_j) \geqslant 1 - \varepsilon$.*

(i) *For $1 \leqslant i \leqslant r$, let W_i be a subset of V_i with $|W_i| \geqslant (1-\delta)m$. Then for every $1 \leqslant i < j \leqslant r$, $d(W_i, W_j) \geqslant 1 - \frac{\varepsilon}{(1-\delta)^2}$.*

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(ii) *If* $\binom{r}{2}\varepsilon < 1$*, then there exist* $v_1 \in V_1, \ldots, v_r \in V_r$ *such that* $v_1 v_2 \cdots v_r$ *forms a copy of* K_r*.*

Proof of Proposition 1.2. To prove (i), by symmetry, it suffices to show that $d(W_1, W_2) \geqslant 1 - \frac{\varepsilon}{(1-\delta)^2}$. Let M be the number of pairs (x, y) such that $xy \notin E(G)$ and $x \in V_1, y \in V_2$. It is clear that

$$(1 - d(W_1, W_2))(1-\delta)^2 m^2 \leqslant (1 - d(W_1, W_2))|W_1||W_2| \leqslant M = (1 - d(V_1, V_2))|V_1||V_2| \leqslant \varepsilon m^2.$$

Rearranging gives the desired lower bound on $d(W_1, W_2)$.

To prove (ii), Let E be the set of all edges produced by the pairs (V_i, V_j), $1 \leqslant i < j \leqslant r$. For $1 \leqslant i \leqslant r$, pick x_i uniformly and independently at random from V_i. It is easy to see that for every $i \neq j$, $\Pr[x_i x_j \notin E] \leqslant \varepsilon$. Then by the union bound, we have

$$\Pr\big[x_1 x_2 \cdots x_r \text{ does not form a copy of } K_r\big] = \Pr\big[\exists\ i \neq j \text{ s.t. } x_i x_j \notin E\big] \leqslant \binom{r}{2}\varepsilon < 1,$$

which implies that with positive probability, one can find a copy of K_r whose r vertices are located in distinct V_i's. □

The following result can be viewed as the fundamental result for set systems with bounded VC-dimension, which was proven in [6, 7, 9].

Lemma 1.3. *Let* $\mathcal{F}$ *be a set system with ground set* V*, and VC-dimension at most* d*, then we have*

$$|\mathcal{F}| \leqslant \sum_{i=0}^{d} \binom{|V|}{i}.$$

Let $0 < \gamma < 1$ be a real number, we say a pair of vertex sets (V_1, V_2) of G is γ-*homogeneous* if the density $d(V_1, V_2) := \frac{|E(V_1,V_2)|}{|V_1||V_2|}$ is either less than γ or larger than $1-\gamma$, where in the former case, we call (V_1, V_2) γ-*sparse*, and in the latter case, we call (V_1, V_2) γ-*dense*. A partition of $V(G)$ is called *equitable* if every two parts differ in size by at most one. We will ultize the following version of regularity lemma. Notice that the work of Łuczak [5] and Goddard and Lyle [3] determined the homomorphism threshold of K_r using the original regularity lemma of Szemerédi [8], which only gave a tower-type upper bound on the size of F. Thus our result in some sense demonstrates the power of the regularity lemma for graphs with bounded VC-dimension.

Theorem 1.4 ([2])**.** *Let* $\varepsilon_0 \in (0, \frac{1}{4})$ *and* $G = (V, E)$ *be an* n*-vertex graph with VC-dimension* d*. Then* $V(G)$ *has an equitable partition* $V(G) = V_1 \cup \cdots \cup V_K$ *with* $\frac{8}{\varepsilon_0} \leqslant K \leqslant c\big(\frac{1}{\varepsilon_0}\big)^{2d+1}$ *parts such that all but an* ε_0*-fraction of the pairs of parts are* ε_0*-homogeneous, where* $c = c(d)$ *is a constant depending only on* d*.*

In particular, the number of parts in the above result of Fox, Pach and Suk is at most $(\frac{1}{\varepsilon_0})^{O(d)}$, which improves the previously known quantitative results by Alon, Fischer and Newman [1] and by Lovász and Szegedy [4]. Moreover, it was shown in [2] that the partition size in the above is tight up to an absolute constant factor in the exponent.

We have also proved that the graph G has bounded VC-dimension.

Claim 1.5. *The VC-dimension of* G *is at most* $t + r - 4$*.*

1.2 Formal proof

Now we are ready to apply Theorem 1.4. Let $\varepsilon_0 = \min\{\varepsilon^{10}, \frac{1}{10r^5}\}$. Note that $\varepsilon_0 \leqslant \frac{1}{r^2} < \frac{1}{4}$ as $r \geqslant 3$. By Theorem 1.4, $V(G)$ has an equitable partition $V(G) = V_1 \cup V_2 \cup \cdots \cup V_K$ with $\frac{8}{\varepsilon_0} \leqslant K \leqslant c(\frac{1}{\varepsilon_0})^{2t+2r-7}$ such that all but an ε_0-fraction of the pairs (V_i, V_j) are ε_0-homogeneous.

Next we claim that there are at most $\sqrt{\varepsilon_0}K$ sets V_i such that there are more than $\sqrt{\varepsilon_0}K$ sets V_j with (V_i, V_j) not being ε_0-homogeneous. Indeed, if otherwise, then the number of pairs of vertex parts which are not ε_0-homogeneous is at least $(\sqrt{\varepsilon_0}K)^2/2 = \varepsilon_0 K^2/2 > \varepsilon_0\binom{K}{2}$, a contradiction. We then move all vertices in those V_i to a new set V_0 and renew $V(G) = V_0 \cup U_1 \cup \cdots \cup U_M$, where $M \leqslant K$ and for every $1 \leqslant s \leqslant M$, $U_s = V_{s'}$ for some $1 \leqslant s' \leqslant K$. Obviously, $|V_0| \leqslant \sqrt{\varepsilon_0}K \cdot \frac{n}{K} = \sqrt{\varepsilon_0} \cdot n$.

Claim 1.6. *For fixed integers $1 \leqslant i, j \leqslant M$, if $d(U_i, U_j) \leqslant 1 - \varepsilon_1$, where $\varepsilon_1 = \frac{4\sqrt{\varepsilon_0}}{\varepsilon}$, then there exist distinct integers $k_1, k_2, \ldots, k_{r-2} \in [M]\backslash\{i, j\}$ such that (V_a, V_b) are ε_0-dense for all $(a, b) \in \binom{\{i,j,k_1,\ldots,k_{r-2}\}}{2}\backslash(i, j)$ if $i \neq j$, and for all $(a, b) \in \binom{\{i,k_1,\ldots,k_{r-2}\}}{2}$ if i=j.*

Proof of claim. For convenience, set $m := \frac{n}{K}$. Let $K_r^-(e)$ be the subgraph obtained from K_r by removing an edge e and $Z(e)$ be the number of copies of $K_r^-(e)$ in G. To prove the claim, let us count $\sum_{u \in U_i, v \in U_j:\ uv \notin E(G)} Z(uv)$. First of all, by assumption, if $i \neq j$, we have that

$$\sum_{u \in U_i, v \in U_j:\ uv \notin E(G)} Z(uv) \geqslant (1 - d(U_i, U_j)) \cdot |U_i||U_j| \varepsilon n^{r-2} \geqslant \varepsilon_1 m^2 \varepsilon n^{r-2}.$$

Suppose that there do not exist distinct integers $k_1, k_2, \ldots, k_{r-2} \in [M]\backslash\{i, j\}$ such that (U_a, U_b) are ε_0-dense for all $(a, b) \in \binom{\{i,j,k_1,\ldots,k_r\}}{2}\backslash(i, j)$. Then for every pair of $u \in U_i, v \in U_j$ with $uv \notin E(G)$, the subgraph $K_r^-(uv)$ belongs to at least one of the following types:

- **Type 1:** At least one vertex of $K_r^-(uv)$ belongs to V_0. Then

$$\sum_{\substack{u \in U_i, v \in U_j:\ uv \notin E(G) \\ \text{Type 1}}} Z(uv) \leqslant m^2 \cdot |V_0| \cdot n^{r-3} \leqslant \sqrt{\varepsilon_0} m^2 n^{r-2},$$

  where the first inequality holds since there are at most m^2 choices for u, v, at most $|V_0| \leqslant \sqrt{\varepsilon_0}n$ choices for a vertex in V_0, and at most n^{r-3} choices for the other $r-3$ vertices.

- **Type 2:** At least two vertices of $K_r^-(uv)$ are contained in the same part U_p. Then

$$\sum_{\substack{u \in U_i, v \in U_j:\ uv \notin E(G) \\ \text{Type 2}}} Z(uv) \leqslant m^2 \cdot (K-2) \cdot \binom{m}{2} \cdot n^{r-4} + m^2 \cdot 2(m-1) \cdot n^{r-3} \leqslant \frac{5}{2} m^3 n^{r-3}.$$

  We briefly explain the first inequality as follows. Since there are at most m^2 choices for u and v, if $p \in [M]\backslash\{i, j\}$, then there at most $K-2$ choices for the part U_p, at most $\binom{m}{2}$ choices for two vertices in U_p, and at most n^{r-4} choices for the remaining $r-4$ vertices; otherwise $p \in \{i, j\}$, there are at most $2(m-1)$ choices for w such that w lies in the part U_i or U_j, and at most n^{r-3} choices for the remaining $r-3$ vertices.

- **Type 3:** At least one edge of $K_r^-(uv)$ is not contained in an ε_0-homogeneous pair. As there are at most $\varepsilon_0\binom{K}{2}$ parts that are not ε_0-homogeneous, we have that

$$\sum_{\substack{u \in U_i, v \in U_j:\ uv \notin E(G) \\ \text{Type 3}}} Z(uv) \leqslant m^2 \cdot \varepsilon_0 \binom{K}{2} m^2 \cdot n^{r-4} + m^2 \cdot 2\sqrt{\varepsilon_0} K m \cdot n^{r-3} \leqslant (2\sqrt{\varepsilon_0} + \varepsilon_0) m^2 n^{r-2}.$$

  The first inequality holds since there are at most m^2 choices for u and v, if the non-ε_0-homogeneous pair does not involve U_i or U_j, then there are at most $\varepsilon_0\binom{K}{2}m^2$ choices for one edge which is not contained in an ε_0-homogeneous pairs, and at most n^{r-4} choices for the remaining $r-4$ vertices; otherwise, there are at most $2\sqrt{\varepsilon_0}Km$ vertices belonging to the parts U_q such that (U_i, U_q) or (U_j, U_q) is not ε_0-homogeneous.

- **Type 4:** At least one edge of $K_r^-(uv)$ is contained in some ε_0-sparse pair. Then

$$\sum_{\substack{u\in U_i, v\in U_j:\ uv\notin E(G)\\ \text{Type 4}}} Z(uv) \leqslant m^2 \cdot \binom{K}{2}\varepsilon_0 m^2 \cdot n^{r-4} + 2\varepsilon_0 m^2 \cdot n \cdot n^{r-3} \leqslant 3\varepsilon_0 m^2 n^{r-2}.$$

We briefly explain the first inequality as follows. If there is at least one edge of $K_r^-(uv)$ is contained in some ε_0-sparse pair and the ε_0-sparse pair does not intersect U_i or U_j, then totally there are at most m^2 choices for u and v, at most $\binom{K}{2}\varepsilon_0 m^2$ choices for one edge contained which is in an ε_0-sparse pair, and at most n^{r-4} choices for the remaining $r-4$ vertices. Otherwise, totally there are at most $2\varepsilon_0 m^2 \cdot n$ triples $(u_i, u_j, u_k) \in U_i \times U_j \times U_k$ such that $u_i u_k u_j$ forms an induced path and (U_i, U_j) is ε_1-sparse and at least one of (U_i, U_k) and (U_j, U_k) is ε_0-sparse, this is because there are at most $\varepsilon_0 m^2$ edges between ε_0-sparse pairs. Finally, the number of choices for the remaining $r-3$ vertices is at most n^{r-3}.

As $\varepsilon_1 = \frac{4\sqrt{\varepsilon_0}}{\varepsilon}$, we have

$$\sum_{u\in U_i, v\in U_j:\ uv\notin E(G)} Z(uv) \geqslant \varepsilon_1 \varepsilon m^2 n^{r-2} > (3\sqrt{\varepsilon_0} + 4\varepsilon_0) m^2 n^{r-2} + \frac{5m^3 n^{r-3}}{2} \geqslant \sum_{u\in U_i, v\in U_j:\ uv\notin E(G)} Z(uv),$$

which leads to a contradiction. Note that the above argument also holds if $i = j$ by hypothesis $d(U_i) < 1 - \varepsilon_1$. Indeed, because G is K_r-free, by Turán's theorem the number of edges in $G[U_i]$ is at most $(1 - \frac{1}{r-1})\binom{m}{2} < (1-\varepsilon_1)m^2$. The proof is finished. ■

By Proposition 1.2 and Claim 1.6, we can refine the original partition by Theorem 1.4 in the following lemma.

Lemma 1.7. *For every integer $r \geqslant 3$, every real number $\varepsilon > 0$ and $t = \lfloor \frac{1}{\varepsilon} \rfloor + 1$, there exists some $\varepsilon_0 = \min\{\varepsilon^{10}, \frac{1}{10r^5}\}$ such that the following holds. Let G be an n-vertex ε-ultra maximal K_r-free graph, then there exists a subset V_0 with $|V_0| \leqslant \sqrt{\varepsilon_0} n$ such that $V(G)\backslash V_0$ can be equitably partitioned into at most $M \leqslant c(t,r)(\frac{1}{\varepsilon_0})^{2(t+r-4)}$ parts $U_1 \cup \cdots \cup U_M$ such that for any distinct $1 \leqslant i < j \leqslant M$, either $d(U_i, U_j) < \varepsilon_2$ or $d(U_i, U_j) > 1 - \varepsilon_1$, where $\varepsilon_1 = \frac{4\sqrt{\varepsilon_0}}{\varepsilon}$ and $\varepsilon_2 = 16r\varepsilon_0$.*

Proof of Lemma 1.7. Based on Claim 1.5 and Claim 1.6, it suffices to show that for any distinct $1 \leqslant i < j \leqslant M$. If $d(U_i, U_j) \leqslant 1 - \varepsilon_1$, then $d(U_i, U_j) < \varepsilon_2$.

Recall $|U_i| = m = \frac{n}{K}$, we also set $C = \frac{1}{8\varepsilon_0}$, and $\varepsilon_2 = 16r\varepsilon_0$. Suppose for the sake of contradiction that there exist two parts, namely U_{r-1} and U_r, such that $\varepsilon_2 \leqslant d(U_{r-1}, U_r) \leqslant 1 - \varepsilon_1$. Then one can find $r-2$ subsets, say $U_1, \ldots, U_{r-2}$, that satisfy the conclusion of Claim 1.6. We will show that if $d(U_{r-1}, U_r) \geqslant \varepsilon_2$, then there must exist a copy of K_r whose r vertices are located in distinct parts $U_1, \ldots, U_r$.

Pick $x \in U_{r-1}$ and $y \in U_r$ uniformly and independently at random. Since by assumption $d(U_{r-1}, U_r) \geqslant \varepsilon_2$, we have $\Pr[xy \in E(G)] \geqslant \varepsilon_2$. To obtain the desired K_r using Proposition 1.2, we will show that with probability larger than $1 - \varepsilon_2$, $|N(x) \cap N(y) \cap U_k|$ are very large for all $k \in [r-2]$. To do so, note first that for every $k \in [r-2]$, $\mathbb{E}\big[|N(x) \cap U_k|\big] \geqslant (1-\varepsilon_0)m$, which implies that $\mathbb{E}\big[|U_k \backslash (N(x) \cap U_k)|\big] \leqslant \varepsilon_0 m$. Therefore, by Markov's inequality we have

$$\Pr\Big[|N(x) \cap U_k| \leqslant (1 - C\varepsilon_0)m\Big] = \Pr\Big[|U_k \backslash (N(x) \cap U_k)| \geqslant C\varepsilon_0 m\Big] \leqslant \frac{1}{C}.$$

By the union bound, it is clear that

$$\Pr\Big[\exists\ k \in [r-2] \text{ s.t. } |N(x) \cap U_k| \leqslant (1 - C\varepsilon_0)m\Big] \leqslant \frac{r-2}{C}.$$

Similarly, one can show that

$$\Pr\left[\exists\ k \in [r-2] \text{ s.t. } |N(y) \cap U_k| \leqslant (1 - C\varepsilon_0)m\right] \leqslant \frac{r-2}{C}.$$

Combining the above two inequalities and the union bound, it is not hard to see that

$$\Pr\left[\forall\ k \in [r-2],\ \min\left\{|N(x) \cap U_k|,\ |N(y) \cap U_k|\right\} \geqslant (1 - C\varepsilon_0)m\right] \geqslant 1 - \frac{2r-4}{C} > 1 - \varepsilon_2.$$

Recall that $\Pr[xy \in E(G)] \geqslant \varepsilon_2$. It follows that there exist $x \in U_{r-1}$ and $y \in U_r$ such that $xy \in E(G)$ and for every $i \in [r-2]$, we have $\min\left\{|N(x) \cap U_k|,\ |N(y) \cap U_k|\right\} \geqslant (1 - C\varepsilon_0)m$. Therefore, for each $k \in [r-2]$, we have $|N(x) \cap N(y) \cap U_k| \geqslant (1 - 2C\varepsilon_0)m$. Let $W_k := N(x) \cap N(y) \cap U_k$. Then it follows by Proposition 1.2 (i) that for every $1 \leqslant k < \ell \leqslant r-2$,

$$d(W_k, W_\ell) \geqslant 1 - \frac{\varepsilon_0}{(1 - 2C\varepsilon_0)^2} = 1 - \frac{16\varepsilon_0}{9}.$$

As $\binom{r-2}{2} \cdot \frac{16\varepsilon_0}{9} \leqslant \frac{8}{9} r^2 \varepsilon_0 < 1$, it follows by Proposition 1.2 (ii) that there exists a copy of K_{r-2} whose $r-2$ vertices are located in distinct W_k's. Together with x, y, we obtain a copy of K_r whose r vertices are located in distinct parts $U_1, \ldots, U_r$, a contradiction. This completes the proof. □

It follows by Lemma 1.7 that for every $1 \leqslant i < j \leqslant M$, (U_i, U_j) is either ε_1-dense or ε_2-sparse. Next, we will make all of the ε_1-dense pairs (U_i, U_j) become complete bipartite graphs by simultaneously destroying all of the missing edges between these two parts.

Let $\mathcal{P}$ be the set formed by all of the missing edges between the ε_1-dense pairs of parts, that is,

$$\mathcal{P} := \{xy \notin E(G) :\ \exists\ 1 \leqslant i < j \leqslant M \text{ s.t. } x \in U_i,\ y \in U_j \text{ and } (U_i, U_j) \text{ is } \varepsilon_1\text{-dense}\}.$$

Let $S = \{x_1y_1, \ldots, x_sy_s\}$ be a maximal matching formed by the missing edges in $\mathcal{P}$. In other words, for each $1 \leqslant \ell \leqslant s$, we have $x_\ell y_\ell \notin E(G)$ and there exist $1 \leqslant \ell_a < \ell_b \leqslant M$ such that $x_\ell \in U_{\ell_a},\ y_\ell \in U_{\ell_b}$ and (U_{ℓ_a}, U_{ℓ_b}) is ε_1-dense; moreover, for every $xy \notin E(G)$ which is a missing edge between some ε_1-dense pair of parts, we must have $\{x, y\} \cap \{x_\ell, y_\ell : 1 \leqslant \ell \leqslant s\} \neq \emptyset$.

To better understand the properties of the maximal matching S, we also need the following new notations. We call an edge $xy \in E(G)$ *sparse* if there exist $1 \leqslant i < j \leqslant M$ such that $x \in U_i,\ y \in U_j$ and (U_i, U_j) is ε_2-sparse. For an integer t, we call a copy of K_t with t vertices locating in t distinct U_i's *sparse* if it contains at least one sparse edge, otherwise we call it *dense*. For each integer $3 \leqslant t \leqslant r-1$, it is not hard to check that the total number of sparse edges and sparse K_t's in G is at most $\varepsilon_2 n^2$ and $\varepsilon_2 n^2 \cdot n^{t-2} = \varepsilon_2 n^t$, respectively.

Claim 1.8. *For every $1 \leqslant \ell \leqslant s$ and $x_\ell y_\ell \in S$, $G[\{x_\ell, y_\ell\} \cap N(x_\ell, y_\ell)]$ contains at least $\frac{1}{3}\varepsilon n^{r-2}$ sparse K_{r-1}'s whose vertex set has non-empty intersection with $\{x_\ell, y_\ell\}$.*

Proof. Recall that by assumption we have $x_\ell y_\ell \notin E(G)$ and there exist $1 \leqslant \ell_a < \ell_b \leqslant M$ such that $x_\ell \in U_{\ell_a},\ y_\ell \in U_{\ell_b}$ and (U_{ℓ_a}, U_{ℓ_b}) is ε_1-dense. Moreover, the induced subgraph $G[N(x_\ell, y_\ell)]$ contains at least εn^{r-2} copies of K_{r-2}'s. Note that the number of K_{r-2}'s with at least two vertices belonging to the same U_i is at most $Km^2 n^{r-4} \leqslant \frac{n^{r-2}}{K}$, and the number of K_{r-2}'s with non-empty intersection with $U_{\ell_a} \cup U_{\ell_b}$ is at most $2mn^{r-3} \leqslant \frac{2n^{r-2}}{K}$. Therefore, by our choice of ε_0 and K, $G[N(x_\ell, y_\ell)]$ contains at least

$$\varepsilon n^{r-2} - \frac{3n^{r-2}}{K} \geqslant \frac{2}{3}\varepsilon n^{r-2}$$

copies K_{r-2}'s whose $r-2$ vertices are located in $r-2$ distinct U_i's, where $1 \leqslant i \leqslant M$ and $i \notin \{\ell_a, \ell_b\}$. It follows that $G[N(x_\ell, y_\ell)]$ contains either at least $\frac{1}{3}\varepsilon n^{r-2}$ dense K_{r-2}'s or at least $\frac{1}{3}\varepsilon n^{r-2}$ sparse K_{r-2}'s, which are disjoint from $U_{\ell_a} \cup U_{\ell_b}$. We then consider the following two cases, depending on the number of dense and sparse K_{r-2}'s in $G[N(x_\ell, y_\ell)]$.

Case 1. Suppose that $G[N(x_\ell, y_\ell)]$ contains at least $\frac{1}{3}\varepsilon n^{r-2}$ dense K_{r-2}'s, which are disjoint from $U_{\ell_a} \cup U_{\ell_b}$. Then note first that as $\binom{r}{2}\varepsilon_1 < 1$, it follows by Proposition 1.2 that G contains no dense K_r. Consider an arbitrary dense K_{r-2} in $G[N(x_\ell, y_\ell)]$, say with vertices $v_1, \ldots, v_{r-2}$, which are disjoint from $U_{\ell_a} \cup U_{\ell_b}$. Then there are distinct integers $1 \leqslant k_1, \ldots, k_{r-2} \leqslant M$, $\{k_1, \ldots, k_{r-2}\} \cap \{\ell_a, \ell_b\} = \varnothing$ such that for every $1 \leqslant t \leqslant r-2$, $v_t \in U_{k_t}$, and all pairs of $U_{k_1}, \ldots, U_{k_{r-2}}$ are ε_1-dense. Since G contains no dense K_r and (U_{ℓ_a}, U_{ℓ_b}) is ε_1-dense, at least one of the $2r-4$ pairs $\{(U_{\ell_a}, U_{k_t}),\ (U_{\ell_b}, U_{k_t}) :\ 1 \leqslant t \leqslant r-2\}$ is ε_2-sparse, which implies that at least one of the $2r-4$ edges $\{x_\ell v_i,\ y_\ell v_i :\ 1 \leqslant t \leqslant r-2\}$ is sparse. Thus, each such dense K_{r-2} produces one copy of sparse K_{r-1} whose vertex set has non-empty intersection with $\{x_\ell, y_\ell\}$.

Case 2. Suppose $G[N(x_\ell, y_\ell)]$ contains at least $\frac{1}{3}\varepsilon n^{r-2}$ sparse K_{r-2}'s, which are disjoint from $U_{\ell_a} \cup U_{\ell_b}$. Then together with x_ℓ and y_ℓ, each such sparse K_{r-2} produces two sparse K_{r-1}'s in G. Therefore, $G[\{x_\ell, y_\ell\} \cup N(x_\ell, y_\ell)]$ contains at least $\frac{1}{3}\varepsilon n^{r-2}$ sparse K_{r-1}'s whose vertex set has non-empty intersection with $\{x_\ell, y_\ell\}$.

Then the proof is finished. □

By Claim 1.8, we can obtain the following upper bound on s.

Corollary 1.9. $s \leqslant \frac{6(r-1)\varepsilon_2 n}{\varepsilon}$.

Proof of claim. Note that $\{x_\ell, y_\ell\} \cap \{x_{\ell'}, y_{\ell'}\} = \varnothing$ for $\ell \neq \ell'$, and each sparse K_{r-1} can lie in at most $r-1$ distinct induced subgraphs $G[\{x_\ell, y_\ell\} \cup N(x_\ell, y_\ell)]$. Therefore, by Claim 1.8, G contains $\frac{1}{r-1} \cdot \frac{s}{2} \cdot \frac{1}{3}\varepsilon n^{r-2}$ distinct sparse K_{r-1}'s. Moreover, we already know that the total number sparse K_{r-1}'s in G is at most $\varepsilon_2 n^2 \cdot n^{r-3} = \varepsilon_2 n^{r-1}$, which implies that $s \leqslant \frac{6(r-1)\varepsilon_2 n}{\varepsilon}$. ∎

Now we move all of these $2s$ vertices in S to V_0, and we know that $|V_0| \leqslant \big(\frac{12(r-1)\varepsilon_2}{\varepsilon} + \sqrt{\varepsilon_0}\big)n$. We also denote $Z_i := U_i \backslash \{x_\ell, y_\ell : 1 \leqslant \ell \leqslant s\}$, $1 \leqslant i \leqslant M$. By the maximality of S, if (U_i, U_j) is ε_1-dense then (Z_i, Z_j) is either complete, or at least one of $Z_i,\ Z_j$ is anti-complete.

To sum up, we have shown that by moving at most $\frac{12(r-1)\varepsilon_2 n}{\varepsilon}$ vertices from $\bigcup_{i=1}^{M} U_i$ to V_0, we can make all of the ε_1-dense pairs (U_i, U_j) become complete. Then we can see $V_0, Z_1, \ldots, Z_M$ form a new partition of $V(G)$ with $|V_0| \leqslant \big(\frac{12(r-1)\varepsilon_2}{\varepsilon} + \sqrt{\varepsilon_0}\big)n$. Let $\varepsilon_3 = \varepsilon^3$. We will slightly refine the above partition by removing all vertices in the Z_i's with $|Z_i| < (1-\varepsilon_3)m$, and decreasing the size of all the remaining Z_j's to $(1-\varepsilon_3)m$ by removing some arbitrary $|Z_j| - (1-\varepsilon_3)m \leqslant \varepsilon_3 m$ vertices. We will put all of the abandoned vertices to V_0. For convenience, we still denote this exceptional set by V_0. Clearly, in the first step of the refinement above, we have removed at most $(1-\varepsilon_3)m \cdot \frac{12(r-1)\varepsilon_2 n/\varepsilon}{\varepsilon_3 m} \leqslant \frac{12(r-1)\varepsilon_2 n}{\varepsilon_3 \varepsilon}$ vertices, and in the second step, we have removed at most $\varepsilon_3 n$ vertices. Finally, we obtain a new partition of $V(G) = V_0 \cup Z_1 \cup Z_2 \cup \cdots \cup Z_J$, where $J \leqslant M \leqslant K$, $|Z_i| = (1-\varepsilon_3)m$ and $|V_0| \leqslant \frac{12(r-1)\varepsilon_2 n}{\varepsilon_3 \varepsilon} + \varepsilon_3 n + \big(\frac{12(r-1)\varepsilon_2}{\varepsilon} + \sqrt{\varepsilon_0}\big)n \leqslant 2\varepsilon^3 n$. Moreover, For each $1 \leqslant i < j \leqslant J$, the pair (Z_i, Z_j) is either complete, or $\frac{\varepsilon_2}{(1-\varepsilon_3)^2}$-sparse.

For each $i \in [J]$, since each $G[Z_i]$ is K_r-free, by Turán's theorem we know that for each Z_i, $e(G[Z_i]) \leqslant \frac{r-2}{r-1}\binom{|Z_i|}{2}$, which implies that there are at least $\frac{1}{r-1}\binom{(1-\varepsilon_3)m}{2} \geqslant \frac{m^2}{16(r-1)} \geqslant \varepsilon_1 m^2$ many missing edges in $G[Z_i]$. Then Claim 1.6 states that for any Z_i, there exist distinct integers $k_1, k_2, \ldots, k_{r-2} \in [J]\backslash\{i\}$ such that (Z_a, Z_b) are ε_1-dense for all $(a,b) \in \binom{\{i, k_1, \ldots, k_{r-2}\}}{2}$. Moreover, since we have proved the ε_1-dense pairs indeed form complete bipartite graphs and G is K_r-free, the following result holds.

Claim 1.10. *For any $1 \leqslant i \leqslant J$, Z_i is an independent set.*

Similarly, suppose that (Z_i, Z_j) is $\frac{\varepsilon_2}{(1-\varepsilon_3)^2}$-sparse, then via a simple modification of the proof of Claim 1.6, we can show there exist distinct integers $k_1, k_2, \dots, k_{r-2} \in [J]\backslash\{i, j\}$ such that (Z_a, Z_b) form complete bipartite graphs for all $(a, b) \in \binom{\{i,j,k_1,\dots,k_{r-2}\}}{2}\backslash(i, j)$. Also by the by the assumption that G is K_r-free, we have the following consequence.

Claim 1.11. *For any Z_i, Z_j with $1 \leqslant i < j \leqslant J$, if (Z_i, Z_j) is not complete, then (Z_i, Z_j) is anti-complete.*

Through the above analysis, now we know that $G[V\backslash V_0] = G[Z_1 \cup \cdots \cup Z_J]$ can be viewed as a $(1 - \varepsilon_3)m$-blow-up of some K_r-free graph. Next, we consider the vertices in V_0.

Claim 1.12. *For any vertex $v \in V_0$ and any Z_i, either $N_{Z_i}(v) = \varnothing$ or $N_{Z_i}(v) = Z_i$.*

Proof of claim. Suppose that there exist two vertices $u_1, u_2 \in Z_i$ such that $vu_1 \in E(G)$ and $vu_2 \notin E(G)$, then by assumption, $G[N(v, u_2)]$ contains $\varepsilon n^{r-2} - |V_0|n^{r-3} \geqslant \frac{\varepsilon n^{r-2}}{2}$ many K_{r-2} such that all of the $r - 2$ vertices are in $Z_1 \cup \cdots \cup Z_J$. By Claim 1.10, these $r - 2$ vertices, say $u_{i_1}, \dots, u_{i_{r-2}} \notin Z_i$ should lie in distinct parts $Z_{i_1}, \dots, Z_{i_{r-2}}$, where $i_x \neq i$ for any $x \in [r - 2]$. Moreover, we know that if there is an edge between any pair (Z_i, Z_j), then this pair (Z_i, Z_j) forms a complete bipartite graph, therefore, $vu_1u_{i_1} \cdots u_{i_{r-2}}$ forms a copy of K_r, a contradiction. ■

Then we partition the vertex set V_0 by the following rule. For any subset $A \subseteq [J]$, we put the vertex $v \in V_0$ into H_A if and only if $N(v)\backslash V_0 = \bigcup\limits_{a\in A} Z_a$. therefore, we partition V_0 into at most 2^J parts, denoted as $V_0 = H_1 \cup H_2 \cup \cdots \cup H_T$, where $T \leqslant 2^J \leqslant 2^K$.

Claim 1.13. *For any pair (H_i, H_j), H_i, H_j is either complete or anti-complete.*

Proof of claim. For any pair H_i, H_j, suppose that there are two pairs of vertices $v_1u_1 \in E(G)$ and $v_2u_2 \notin E(G)$, then by assumption that $G[N(v_2, u_2)]$ contains $\varepsilon n^{r-2} - |V_0|n^{r-3} \geqslant \frac{\varepsilon n^{r-2}}{2}$ many K_{r-2}'s such that all of the $r - 2$ vertices are in $Z_1 \cup \cdots \cup Z_J$. By Claim 1.10, these $r - 2$ vertices, namely, $u_{i_1}, \dots, u_{i_{r-2}}$ should lie in distinct parts $Z_{i_1}, \dots, Z_{i_{r-2}}$. By Claim 1.12, we know all of the pairs (H_a, Z_b) form complete bipartite graphs for all $a \in \{i, j\}$ and $b \in \{i_1, \dots, i_{r-2}\}$. Therefore, $v_1u_1u_{i_1} \cdots u_{i_{r-2}}$ forms a copy of K_r, a contradiction. ■

Trivially the above argument gives that $L \leqslant 2^K + K \leqslant 2^{c(\frac{1}{\varepsilon_0})^{2t+2r-7}}$, as $\varepsilon_0 = \min\{\varepsilon^{10}, \frac{1}{10r^5}\}$, $t = \lfloor\frac{1}{\varepsilon}\rfloor + 1$ and c only depends on ε and r. More carefully, we can further improve the estimation. By the role of partition of V_0, we consider the set system

$$\mathcal{F} := \{F \subseteq [J] : \text{there is some vertex } v \in V_0 \text{ such that } N(v)\backslash V_0 = \bigcup_{j\in F} Z_j\}.$$

Suppose that $|\mathcal{F}| > \sum\limits_{i=0}^{t+r-4} \binom{|J|}{i}$, then Lemma 1.3 tells that there is a subset $A \subseteq [J]$ with $|A| = t + r - 3$ such that A is shattered by $\mathcal{F}$. We can pick one vertex from each Z_i with $i \in A$ respectively, say $a_1, a_2, \dots, a_{t+r-3}$. By definition of $\mathcal{F}$, for any subset $B \subseteq \{a_1, a_2, \dots, a_{t+r-3}\}$, we can find some vertex v_B in V_0 such that $N(v_B) \cap \{a_1, a_2, \dots, a_{t+r-3}\} = B$, which implies that the VC-dimension of G is at least $t + r - 3$, a contradiction to Claim 1.5. Therefore, $|\mathcal{F}| \leqslant \sum\limits_{i=0}^{t+r-4} \binom{|J|}{i}$, which means $T \leqslant \sum\limits_{i=0}^{t+r-4} \binom{|J|}{i}$. Therefore, $L \leqslant J + \sum\limits_{i=0}^{t+r-4} \binom{|J|}{i} = 2^{c\left(\frac{1}{\varepsilon}+r\right)^2 \log\max\left\{\frac{1}{\varepsilon^{10}}, 10r^5\right\}}$ for some absolute constant $c > 0$. This finishes the proof.

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