Multi-color Ramsey numbers of K_4 versus K_t

Abstract

Based on a recent breakthrough on Ramsey number of $R(K_4, K_t)$ and a standard application of random blowup methods, we can show that $R_k(K_4; K_t) = \tilde{\Theta}(t^{2k+1})$.

1 Main result

Let $R_k(H; K_t)$ be the minimum number N such that for any $(k+1)$-coloring of K_N, there is no H in each color class i, $1 \leqslant i \leqslant k$, and no K_t in color class $k+1$. A recent breakthrough of Mattheus and Verstraëte [4] showed that $R_1(K_4; K_t) \geqslant \Omega(\frac{t^3}{\log^4 t})$, matches the upper upper bound in [1] up to a factor of order $\log^2 t$. Based on this and the blowup method introduced by Alon and Rödl [2], we give a simple consequence and the upper bound $O(\frac{t^{2k+1}}{\log^{2k} t})$ can be found in [3].

Theorem 1.1. $R_k(K_4; K_m) = \tilde{\Theta}(m^{2k+1})$.

Proof. Let t be a sufficiently large integer, $n = \frac{c_1 t^3}{\log^4 t}$, $r = c_2 t^{2k-2} \log^a t$, $m = t \log t$. One can check that $t\binom{N}{t}\binom{rt}{m} < \binom{rN}{m}^{(k-1)/k}$ by suitable choices of c_1, c_2 and a. Let H be an n-vertex K_4-free graph with $\alpha(H) < t$ in [4]. Let G be an r-blowup of H, so $|V(G)| := N = rn$ and G is also K_4-free. We randomly embed k copies of G into an N-set and denote them as $G_1, G_2, \ldots, G_k$. Let G_{k+1} be the graph on the same N-set and the edge set of G_{k+1} consists of all pairs which do not belong to any $E(G_i)$ for $1 \leqslant i \leqslant k$. We color K_N in the following way, for an edge $e \in E(K_N)$, we color it i if i is the minimum index such that $e \in E(G_i)$ and color it $k+1$ if there is no such G_i.

For each $1 \leqslant i \leqslant k$, there is no K_4 in the color class i. Let M be the number of independent sets of size m in G, note that M is at most $\sum_{t' \leqslant t} \binom{N}{t'}\binom{rt'}{m} \leqslant t\binom{N}{t}\binom{rt}{m}$, because each vertex in K_m of G_{k+1} should come from some independent set in H and $\alpha(H) < t$. Then for any m-subset in K_N, the probability that it forms an K_m in G_{k+1} is $(\frac{M}{\binom{rN}{m}})^k$, and the expected number of K_m is $\binom{rN}{m}(\frac{M}{\binom{rN}{m}})^k < 1$. Therefore there exist some suitable mappings such that $G_1, G_2, \ldots, G_k$ are K_4-free and G_{k+1} is K_m-free, which gives $R_k(K_4; K_m) \geqslant c_1 c_2 m^{2k+1} / \log^{a-4} t$. $\square$

References

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