Yesterday

- Friedgut–Kahn
- Mixture bound
  - KK
  - Induced C_5

Rainbow $\triangle$

Colorful ver. of KK:

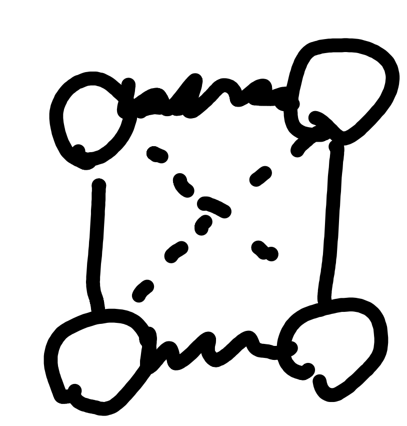
Q R red, G green, B blue edges. max # rainbow $\triangle$?

Thm (C., Yu)

$$T^2 \le 2RGB$$

T

tight:



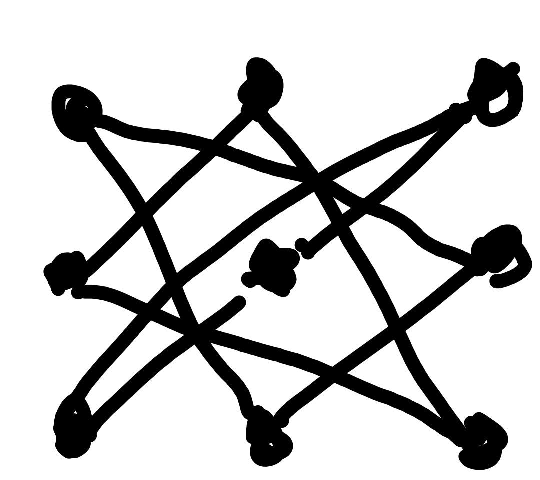
Thm (C., Sankar, Yu)

r colors, m edges (in total), T rainbow $\triangle$

$$T \le \sqrt{\frac{2(r-2)}{9r} m^3}$$

tight: Affine plane of order $r-1$. color by slope

Ex $\mathbb{F}_3^2$



one color

slope $= 1 = -\frac{1}{2}$

Thm (CSY)

Structure and stability of

Thm (Balogh, Bradshaw, Garcia, Lidický)

R red G green B blue

properly colored $K_4 \le \frac{1}{4}(RGB)^{\frac{2}{3}}$

tight

Thm (CSY)

r color, m edge

$\#$ properly colored $K_d \le \binom{r+1}{d}\left(\frac{m}{\binom{r+1}{2}}\right)^{d/2}$

$\boxed{\text{Tight: partition } K_{r+1} \text{ into } r \text{ perfect matchings}}$

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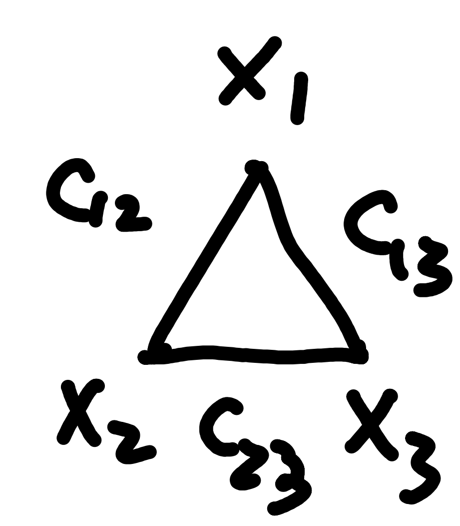
pf of RT w/ r colors

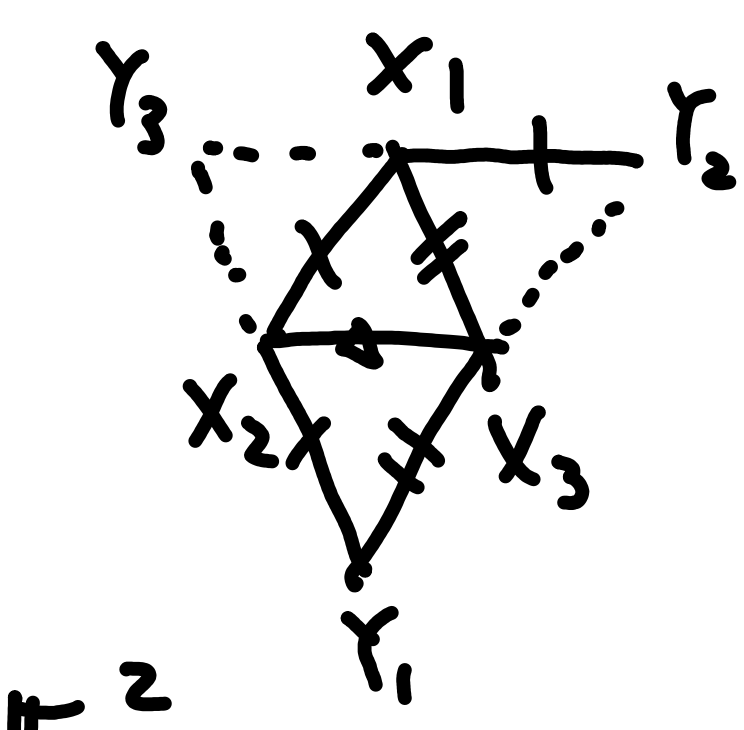
Sample $(X_1, X_2 X_3)$: rainbow $\triangle$ u.a.r.



$H(X_1 X_2 X_3) = \log_2(6T)$

$H(X_1 X_2) \le \log_2(2m)$

Goal $2H(X_1 X_2 X_3) \le 3H(X_1, X_2) + \log_2 \frac{r-2}{r}$

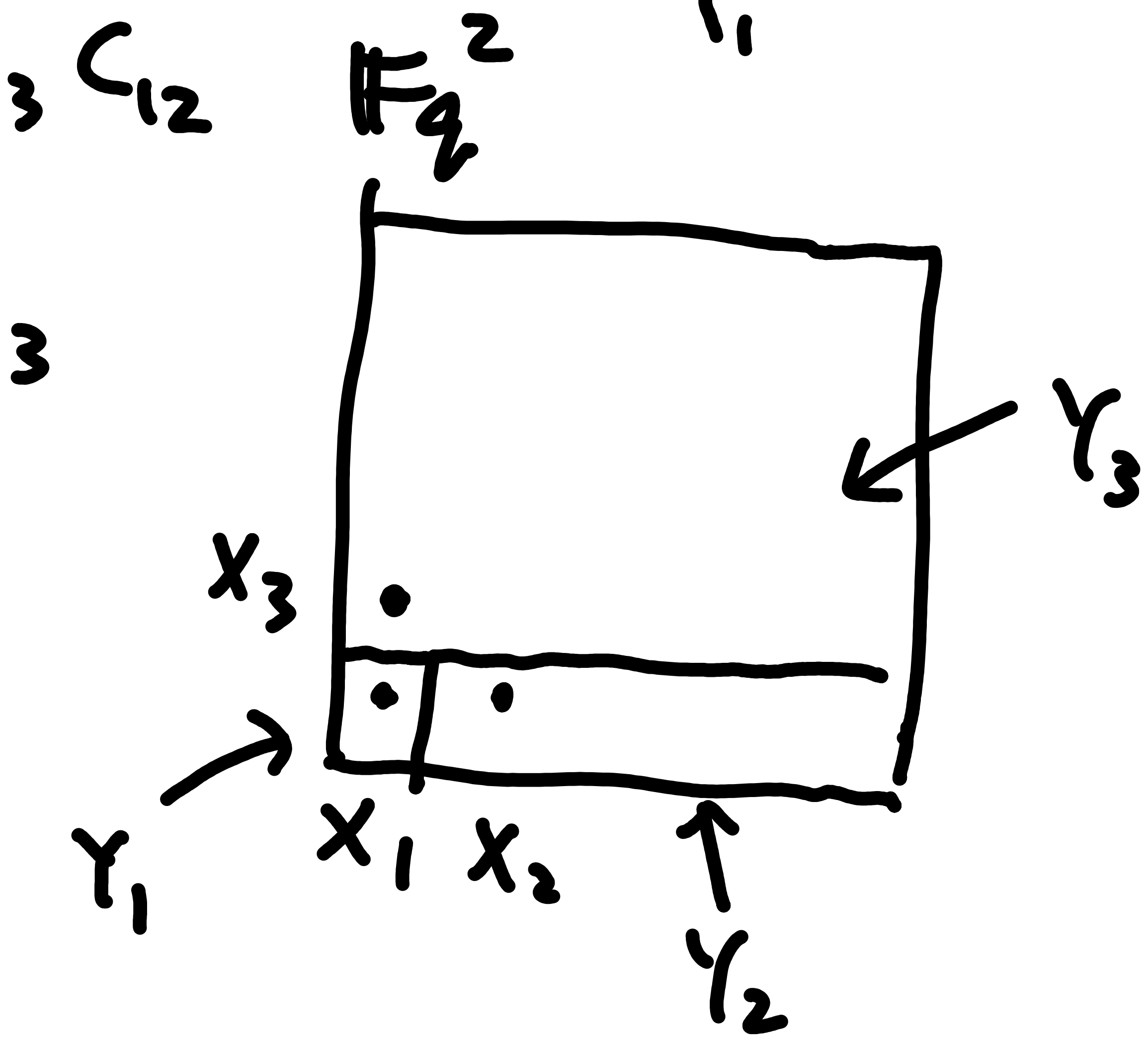


Sample : $Y_1 \mid X_2 X_3, c_{12}, c_{13} \sim X_1 \mid X_2 X_3\, c_{12}\, c_{13}$

CI

$Y_2 \mid X_1 X_3\ c_{12} \sim X_2 \mid X_1\, X_3\, c_{12}$

$Y_3 \mid X_1, X_3 \sim X_3 \mid x_1 X_3$



Claim 1 $H(X_3 \mid X_1 X_2) \le H(X_2 \mid X_1) + \log_2 \frac{r-1}{r}$

Claim 2 $H(X_3 \mid X_1 X_2) \le H(X_1) + \log_2 \frac{r-2}{r-1}$

$\boxed{\text{Claim } 1+2 \Rightarrow \text{thm}}$

<u>Claim 0</u> $(X_1 X_2 X_3 Y_i)$ $i=1,2,3$ have disjoint supp.

<u>pf</u> (I'll hand-wave and dance on stage
Check yourself)

<u>Claim -1</u>
$$H(Y_1|X_1X_2X_3) \geq H(X_3|X_1X_2) - \log_2((r-1)(r-2))$$
$$H(Y_2|X_1X_2X_3) \geq H(X_3|X_1X_2) - \log_2(r-1)$$
$$H(Y_3|X_1X_2X_3) = H(X_3|X_1X_2)$$

<u>pf</u>
$$H(Y_1|X_1X_2X_3) = H(X_1|X_2X_3C_{12}C_{13})$$
$$= \underbrace{H(X_1C_{12}C_{13}|X_2X_3)}_{= H(X_3|X_1X_2)} \underbrace{- H(C_{12}C_{13}|X_2X_3)}_{\geq -\log_2((r-1)(r-2))}$$

$$H(Y_2|X_1X_2X_3) = H(X_2|X_1X_3C_{12})$$
$$= \underbrace{H(X_2C_{12}|X_1X_3)}_{= H(X_3|X_1X_2)} \underbrace{- H(C_{12}|X_1X_3)}_{\geq -\log_2(r-1)}$$

$$H(Y_3|X_1X_2X_3) = H(X_3|X_1X_2) \text{ by def}$$

<u>pf claim 1</u> Mix $(X_1X_2X_3Y_2)$, $(X_1X_2X_3Y_3)$ $\rightsquigarrow (X_1X_2X_3A)$
$$2^{H(X_1X_2X_3Y_2)} + 2^{H(X_1X_2X_3Y_3)} \leq 2^{H(X_1X_2X_3A)}$$
$$\geq \qquad\qquad\qquad \leq$$
$$\left(1+\frac{1}{r-1}\right) 2^{H(X_1X_2X_3)+H(X_3|X_1X_2)} \qquad 2^{H(X_1X_2X_3)+H(A|X_1)}$$

$$\frac{r}{r-1} 2^{H(X_3|X_1X_2)} \le 2^{H(A|X_1)} \qquad Y_2|X_1 \sim X_2|X_1$$
$$= 2^{H(X_2|X_1)} \qquad Y_3|X_1 \sim X_3|X_1$$

$\square$

Pf of Claim 2

Mix $(X_1 X_2 X_3 Y_i)$ $i = 1, 2, 3$

$$\sum_{i=1}^{3} 2^{H(X_1X_2X_3Y_i)} \le 2^{H(X_1X_2X_3B)} \le 2^{H(X_1X_2X_3)+H(B)}$$

$$\vee\!/$$

$$\left(1+\frac{1}{r-1}+\frac{1}{(r-1)(r-2)}\right) 2^{H(X_1X_2X_3)+H(X_3|X_1X_2)} \qquad Y_1 \sim Y_2 \sim Y_3 \sim X_i$$

$$\Rightarrow \frac{r-1}{r-2} 2^{H(X_3|X_1X_2)} \le 2^{H(X_1)}$$

$\square$

Turán

Thm (Density Turán)

$t(K_{r+1}, G) = 0 \Rightarrow t(K_2, G) \le 1 - \frac{1}{r}$

Thm (C., Yu)

Entropy ver: random var. $(X_1 X_2) \in E(G)$ s.t. $(X_1X_2) \sim (X_2X_1)$, then

$$H(X_1, X_2) \le 2H(X_1) + \log_2\left(1-\frac{1}{r}\right)$$

pf w/o entropy: (Caro-Wei essentially)

Pick

$v_0 \; v_1 \; v_2 \; \cdots \in V(G)$ u.a.r.

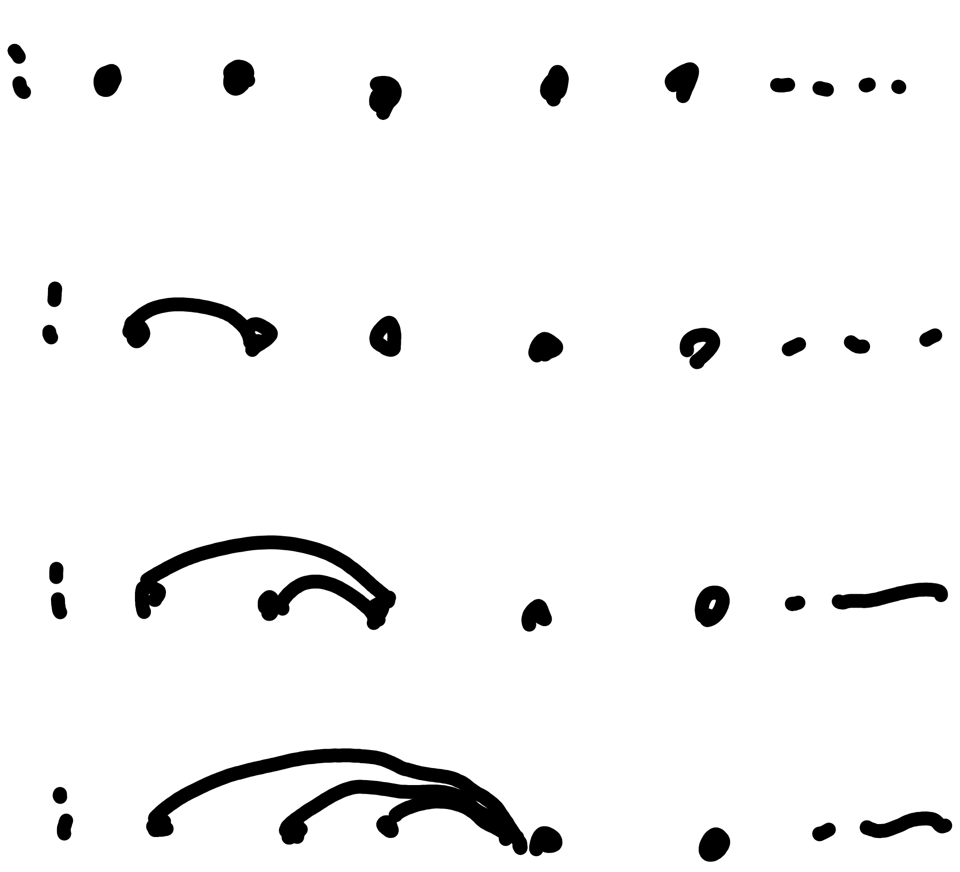
Events:

A_0 :

A_1 :

A_2 :

A_3 :



<u>Claim</u> At most r events happen at the same time

<u>pf</u> $A_0\ A_{i_1}, \cdots, A_{i_r}$: $(r+1)$ events

$\Rightarrow V_0, V_{i_1}, \cdots V_{i_r}$ is a clique $\times$

$\Rightarrow P(A_0) + P(A_1) + \cdots \leq r$

- $P(A_i) = t(K_{1,i}, G) \geq t(K_2, G)^i$ <u>Sidorenko</u>

$$\Rightarrow \frac{1}{1 - t(K_2, G)} = \sum_{i=0}^{\infty} t(K_2, G)^i \leq r \qquad \square$$

<u>pf w/ mix</u>

Sample :

T_0 : $X_1 \;\; X_1 \;\; X_1 \; \cdots \; X_1$

$\sim$

T_1 : $X_2 | X_1 \;\; X_1 \;\; X_1 \; \cdots \; X_1$

T_2 : $X_2|X_1 \quad X_2|X_1 \quad X_1 \quad \cdots \quad X_1$

$\vdots$

<u>Claim</u> $T_0, \cdots, T_{N-1}$ have $(r+1)$-wise disjoint supp.

<u>Pf</u> Same as before

$$\Rightarrow \quad 2^{H(T_0)} + 2^{H(T_1)} + \cdots + 2^{H(T_{N-1})} \leq r \cdot 2^{H(T_i)} \leq r \cdot 2^{N H(X_1)} \quad \cdots (1)$$

$$2^{H(T_i)} = 2^{i H(X_2|X_1) + (N-i) H(X_1)}$$

Set $x = 2^{H(X_2|X_1) - H(X_1)}$

$$(1) \Rightarrow \quad 1 + x + \cdots + x^{N-1} \leq r$$

Take $N \to \infty$: $\frac{1}{1-x} \leq r \Rightarrow x \leq 1 - \frac{1}{r}$

$$H(X_2|X_1) \leq H(X_1) + \log_2\left(1 - \frac{1}{r}\right) \quad \square$$

<u>Thm</u> $\max H(X_1 X_2) - 2H(X_1) = L(G)$ Lagrangian

$\max H(X_1 X_2) - H(X_1) = \rho(G)$ spectral radius

Thus, entropy turan $\Leftrightarrow$ density turan

<u>Cor</u>

$$\rho(G)^k \leq \left(1 - \frac{1}{r}\right) \hom(T, G) \quad \forall \text{ tree } T \text{ w/ } k \text{ vtxs.}$$

($T = \cdot$: Wilf , $T =$ path : Nikiforov)

Some connections w/ incidence geo

HW 3.8: $p \in \mathbb{R}^3$ random point

$l_1, l_2, l_3 \ni p$ random non-coplanar lines

Then $$2H(p) + \sum_{i=1}^{3} H(l_i \mid p) \le \sum_{i=1}^{3} H(l_i) + \log_2 6$$

Def p is a joint of $l_1 l_2 l_3$ if $p \in l_1 \cap l_2 \cap l_3$ and $l_1 l_2 l_3$ non-coplanar

Q Max # joints given m lines?

Ans $O(m^{\frac{3}{2}})$

pf L : m lines in $\mathbb{R}^3$, J : set of joints formed by L

Take p unif from J, $l_1 l_2 l_3$ makes p a joint

$$2\log_2 |J| = 2H(p) \le \sum_{i=1}^{3} H(l_i) + \log_2 6 \le 3\log_2 m + \log_2 6$$

$\Rightarrow |J| = O(m^{\frac{3}{2}})$

Truth : Thm (C. Yu) If $L = \binom{x}{2} \Rightarrow J \le \binom{x}{3}$

$\Rightarrow$ HW 4.9 :

$\mathcal{F} \subseteq \binom{[n]}{t}$, $\mathcal{G} \subseteq \binom{[n]}{t-1}$

$\forall A \in \mathcal{F}$, $\exists B_1, \cdots, B_d \in \mathcal{G}$ s.t. $B_1, \cdots, B_d \subseteq A$

$|\mathcal{F}| = \binom{x}{d} \Rightarrow \mathcal{G} \ge \binom{x}{d-1}$

(WRONG for matroid!!)

Pf poly method

Q Any pf of 3.8 or 4.9 w/o poly?