Yesterday

Applications of Shearer's ineq:

- Generalized Hölder
- Counting indep sets / colorings
- Δ-intersecting family of graphs
- Kruskal-Katona (weak ver) and

$\max \hom(C_5, G) = \Theta(m^{\frac{5}{2}})$ if G has m edges

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$\max \hom(H,G)$ among all G w/ m edges?

Thm (Alon for graphs, Friedgut-Kahn for hypergraphs)

Given H: r-unif hypergraph, w/o iso. vtx.s $\max\limits_{G: m \text{ edges}} \hom(H,G) = \Theta(m^{\alpha^*(H)})$

$\alpha^*(H)$: fractional (edge) covering number

$$= \min \sum_{e \in E(H)} t_e$$

subj. $\sum_{e \ni v} t_e \geq 1 \quad \forall v \in V(H)$

$t_e \geq 0 \quad \forall e \in E(H)$

Pf Sample $X_H = (X_v)_{v \in V(H)} \in \mathrm{Hom}(H,G)$ u.a.r.

Want to upper bound $H(X_H)$ by $H(X_e)$, $e \in E(H)$

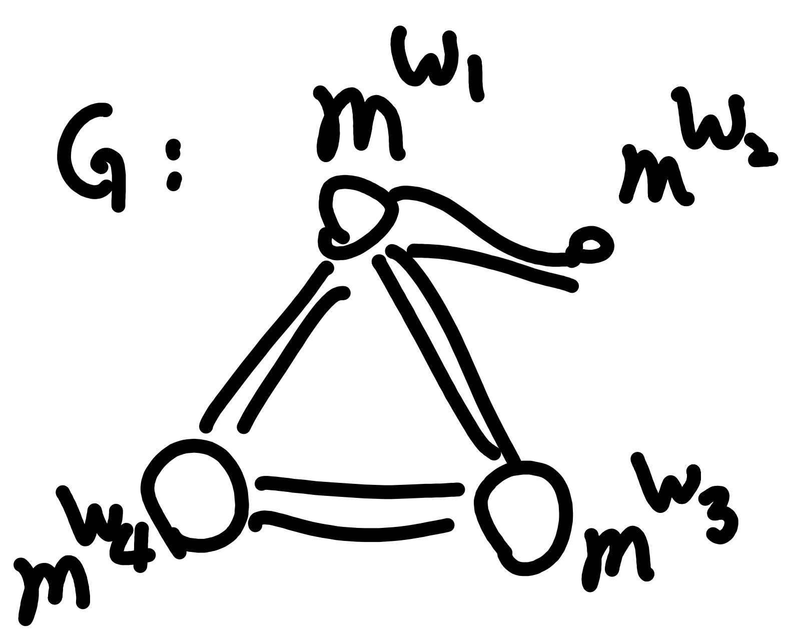
Shearer (weighted ver):

$$\underset{\log_2 \hom(H,G)}{\overset{\shortparallel}{H(X_H)}} \le \sum_{e\in E(H)} t_e H(X_e) \le \Big(\underbrace{\sum_{e\in E(H)} t_e}_{\text{best}\,:\,\alpha^*(H)}\Big) \log_2(r!\, m)$$

$\Rightarrow \hom(H,G) = O(m^{\alpha^*(H)})$

Construction

Idea What's the best one can get from blowups of H

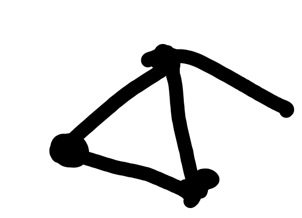
G: 

Req $w_1 + w_2 \le 1$

$w_1 + w_3 \le 1$

$\vdots$

#H $m^{w_1+w_2+w_3+w_4}$



Pf Consider $\max \sum_{v\in V(H)} w_v$ fractional stable number

subj $\sum_{v\in e} w_v \le 1 \quad \forall e \in E(H)$

$w_v \ge 0$

$G = (m^{w_v})_{v\in V(H)}$ blowup of H, $\hom(H,G) = \Omega(m^{\rho^*(H)})$

#edges in $G = O(m)$

Claim This is also $\alpha^*(H)$

Pf Dual LP ! $\square$

Kruskal-Katona (again)

Can we prove the Lovász ver?

$\#\Delta = \binom{x}{3} \Rightarrow m \geq \binom{x}{2}$

Recall

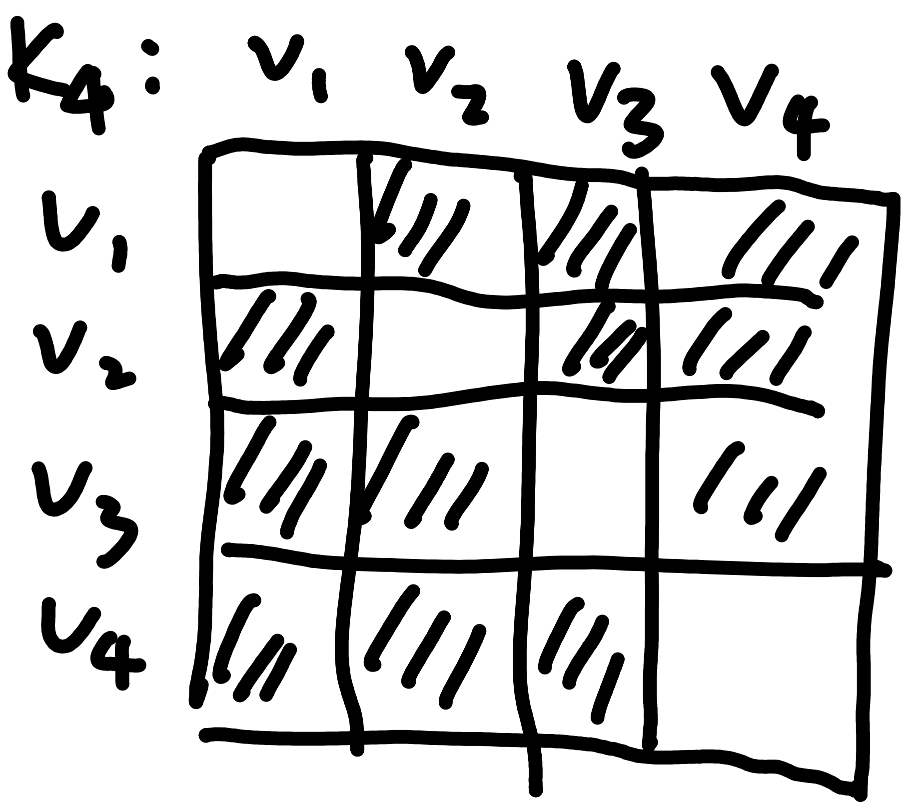
$$2\log_2(6\#\Delta) = \underset{2\log_2(x(x-1)(x-2))}{2H(X_1X_2X_3)} \underset{\text{Shearer}}{\leq} \underset{3\log_2(x(x-1))}{3H(X_1X_2)} \leq 3\log_2(2m)$$

Shearer not tight!

Issue: In Shearer, we used $H(X_1|X_2) \leq H(X_1)$

But X_1, X_2 are not indep. ($X_1 \neq X_2$ always)

$G = K_4$:



(X_1, X_2) unif on off-diagonal

(X_1, X_1) unif on diagonal

Idea Mix (X_1, X_2) w/ (X_1, X_1) to get $(X_1, ?)$ indep.

Def Given $X_1, \dots, X_n$, a mixture Z is a random var $Z \sim X_i$, where $i \in [n]$ is a indep. random index

Equivalently, the pdf of Z is a convex combination of pdf.s of $X_1, \dots, X_n$

Lemma (Mixture bound, C., Y.)

Suppose $\mathrm{supp}(X_1), \dots, \mathrm{supp}(X_n)$ are $(k+1)$-wise disjoint

i.e. no x in $(k+1)$ of them.

Then $\exists$ a mixture Z of $X_1, \dots, X_n$ s.t.

$$2^{H(X_1)} + \dots + 2^{H(X_n)} \le k\, 2^{H(Z)}$$

Ex If $X_1, \dots, X_n$ uniform on $S_1, \dots, S_n$, disjoint

$$k=1: \quad 2^{H(X_1)} + \dots + 2^{H(X_n)} = |S_1| + \dots + |S_n| = 2^{H(Z)}$$

Z: unif on $\bigcup S_i$

Pf Set $s_i = 2^{H(X_i)}$, take $p_i = P(\mathbf{i} = i) = \dfrac{s_i}{\sum_j s_j}$

$$H(Z) = H(X_{\mathbf{i}}) = H(X_{\mathbf{i}}, \mathbf{i}) - H(\mathbf{i} \mid X_{\mathbf{i}})$$

$$= H(X_{\mathbf{i}} \mid \mathbf{i}) + H(\mathbf{i}) - H(\mathbf{i} \mid X_{\mathbf{i}})$$

$$H(\mathbf{i} \mid X_{\mathbf{i}}) \le \log_2 k$$

$$\begin{aligned} H(X_{\mathbf{i}} \mid \mathbf{i}) &= \mathbb{E}_i H(X_{\mathbf{i}} \mid \mathbf{i} = i) \\ &= \mathbb{E}_i H(X_i) \\ &= \sum_i p_i \log_2 s_i \\ &= \sum_i p_i \log_2 p_i + \sum_i p_i \log_2 \sum_j s_j \\ &= -H(\mathbf{i}) + \log_2 \sum_j s_j \end{aligned}$$

pf (kk)

Again, (X_1, X_2, X_3) : u.ar. $\triangle$

Claim $2^{H(X_3|X_1X_2)} + 1 \leq 2^{H(X_2|X_1)}$

$2^{H(X_2|X_1)} + 1 \leq 2^{H(X_1)}$

pf Mix (X_1, X_2) w/ (X_1, X_1)

disjoint supp

$\Rightarrow \exists i$ s.t.

$$2^{H(X_1X_2)} + 2^{H(X_1X_1)} \leq 2^{H(X_1, X_i)} \leq 2^{2H(X_1)}$$

$$\Rightarrow \quad 2^{H(X_2|X_1)} + 1 \leq 2^{H(X_1)}$$

Similarly, mix $(X_1X_2X_3)$, $(X_1X_2X_2)$

$$2^{H(X_1X_2X_3)} + 2^{H(X_1X_2X_2)} \leq 2^{H(X_1X_2, X_{i'})}$$

$$\leq 2^{H(X_1X_2) + H(X_2|X_1)}$$

$$H(X_1X_2X_{i'}) = H(X_{i'}|X_1X_2) + H(X_1X_2)$$

$$X_2|X_1 \sim X_3|X_1 \sim X_{i'}|X_1$$

$$\Rightarrow 2^{H(X_3|X_1X_2)} + 1 \leq 2^{H(X_2|X_1)}$$

Cor If $H(X_3|X_1X_2) = \log_2 r$

then $H(X_2|X_1) \geq \log_2(r+1)$

$H(X_1) \geq \log_2(r+2)$

$$H(X_1 X_2 X_3) = \log_2(x(x-1)(x-2))$$
$$= H(X_1) + H(X_2 | X_1) + H(X_3 | X_1 X_2)$$
$$\geq \log_2((r+2)(r+1)r)$$

$\Rightarrow r \leq x-2$

$$H(X_1 X_2) \geq \log_2((r+2)(r+1)) \geq \log_2(x(x-1)) \quad \square$$

Remark w/ this method, one can prove

- KK for d-unif
- q-KK
- multidim KK

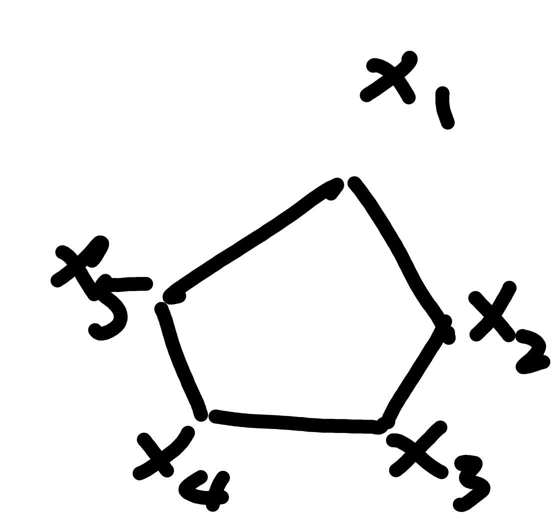
(see HW)

Induced C_5

Thm $\hom_{ind}(C_5, G) \leq \frac{1}{\sqrt{250}}(2m)^{\frac{5}{2}}$

(If G: balanced blowup of C_5, $\hom_{ind}(C_5, G) = \frac{1}{\sqrt{1000}}(2m)^{\frac{5}{2}}$)

Take $X_1, \dots, X_5$: induced C_5 in G u.a.r.



WTS $2H(X_1, \dots, X_5) + \log_2 250 \leq 5H(X_1 X_2)$

Claim $H(X_1 | X_2 \dots X_5) \leq H(X_1 | X_{235}) \leq H(X_1 | X_2) - \log_2 2$

pf Mix $(X_1 X_2 X_3 X_4)$ w/ $(X_1 X_2 X_4 X_3)$

disjoint supp $\rightsquigarrow (X_1 X_2 A B)$

$$2 \cdot 2^{H(X_1 X_2 X_3 X_4)} \leq 2^{H(X_1 X_2 A B)}$$

$$(X_3 X_4, X_1) \sim (X_4 X_3, X_1) \sim (AB, X_1)$$

$$\Rightarrow H(X_1 X_2 AB) \leq H(X_1 AB) + H(X_2 | X_1)$$

$$= H(X_{134}) + H(X_2 | X_1)$$

$$\Rightarrow \underbrace{H(X_2 | X_{134})}_{\substack{\shortparallel \\ H(X_1 | X_{235})}} \leq H(X_2 | X_1) - \log_2 2$$

□

Claim 2

$$H(X_1 | X_{234}) \leq H(X_1) - \log_2 5$$

pf Resample $Y_i | X_{i+1} \cdots X_{i+3} \sim X_i | X_{i+1} \cdots X_{i+3}$

- $(X_1 \cdots X_5, Y_i)$ $i = 1, \ldots, 5$ have disjoint supp.s

pf Say $(x_1, \ldots, x_5, y) \in \mathrm{supp}(X_1, \ldots, X_5, Y_a) \cap \mathrm{supp}(X_1 \cdots X_5 Y_b)$

WLOG $a=1$ $b=2$ or $a=1$ $b=3$

$\boxed{a=1}$: $y x_2 \in E$ $y x_3^{\star}, y x_4^{\triangle} \notin E$

$\boxed{b=2}$: $y x_3^{\star} \in E$ $y x_4$ $y x_5 \notin E$

$\boxed{b=3}$: $y x_4^{\triangle} \in E$ $y x_5$ $y x_1 \notin E$ $\lightning$

$$\Rightarrow 5 \cdot 2^{H(X_1 \cdots X_5 Y_1)} \leq 2^{H(X_1 \cdots X_5) + H(X_1)}$$

$$H(X_1 \cdots X_5 Y) = H(X_1 \cdots X_5) + \underbrace{H(Y_1 | X_2 \cdots X_5)}_{\substack{\shortparallel \\ H(X_1 | X_{234})}}$$

□

$$2H(X_1 \cdots X_5) = 2H(X_1 X_2) + 2\underbrace{H(X_3 | X_1 X_2)}_{1} + 2\underbrace{H(X_4 | X_{123})}_{2}$$

$$+ 2\underbrace{H(X_5 | X_{1 \cdots 4})}_{3}$$

$1 \leq H(X_1 | X_2)$

$3 \leq H(X_1 | X_2) - \log_2 2$ ... Claim 1

$3 \leq 2 \leq H(X_1) - \log_2 5$ ... Claim 2

$$2 \cdot 1 + 3 + (2 \cdot 2 + 3)$$

$$\leq H(X_1 | X_2) + H(X_1 | X_2) - \log_2 2 + 3H(X_1) - \log_2 125$$

$$= 3H(X_1 X_2) - \log_2 250$$

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<u>Issue</u> $H(X_3 | X_1 X_2) \leq H(X_1 | X_2)$ off by a factor of 2 !