Yesterday

- Sidorenko : $t(H,G) \geq t(K_2,G)^{|E(H)|}$

  P_2, P_3, C_4, Q_n Sidorenko

- $t(T,G) \leq t(K_{1,k-1},G)$ $\forall$ k vtx tree T

- Shearer :

  $A_1, \ldots, A_m$ covers $[n]$ k times

  then $k H(X_{[n]}) \leq \sum H(X_{A_i})$

Applications of Shearer's ineq

Thm (Loomis-Whitney / generalized Hölder)

Let $S \subseteq \mathbb{Z}^n$ finite, $A_1, \ldots, A_m \subseteq [n]$ covers i $\geq k$ times $\forall i$

$P_{A_i} : \mathbb{Z}^n \to \mathbb{Z}^{A_i}$ projection

then $|S|^k \leq \prod_{i=1}^{m} |P_{A_i}(S)|$

pf Sample $(x_1, \ldots, x_n) \in \mathbb{Z}^n$ uniformly from S

$$k \log_2 |S| = k H(X_1, \ldots, X_n) \leq \sum H(X_{A_i})$$

$$\leq \sum \log_2 |\mathrm{supp}(X_{A_i})|$$

$$= \log_2 \prod |P_{A_i}(S)|$$

Generalized Hölder (w/ weight)

Thm $A_1, \cdots, A_m \subseteq [n]$, covers $[n]$ at least k times.

$f_i : \mathbb{Z}^{A_i} \to \mathbb{R}_{\geq 0}$, $\pi_{A_i} : \mathbb{Z}^n \to \mathbb{Z}^{A_i}$ proj.

$$\sum_{x \in \mathbb{Z}^d} \prod_i f_i(\pi_{A_i}(x)) \leq \prod_i \left(\sum_{y \in \mathbb{Z}^{A_i}} f_i(y)^k \right)^{\frac{1}{k}}$$

pf Sufficient to show for $f_{A_i} : \mathbb{Z}^{A_i} \to \mathbb{Z}$

Sample $(X_1, \cdots, X_n, Y_1, \cdots, Y_m) \in \mathbb{Z}^{n+m}$ s.t.

$0 \leq Y_i \leq f_i(\pi_{A_i}(X_1, \cdots, X_n))$ u.a.r.

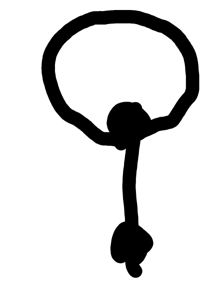
$$\begin{aligned} k \log_2 LHS &= kH(X_1, \cdots, X_n, Y_1, \cdots, Y_m) \\ &= kH(X_1, \cdots, X_n) + kH(Y_1, \cdots, Y_m \mid X_1, \cdots, X_n) \\ &\leq \sum H(X_{A_i}) + \sum k H(Y_i \mid X_{A_i}) \\ &= \sum_i H(X_{A_i}, Y_i^{(1)}, \cdots, Y_i^{(k)}) \end{aligned}$$

condition on X_{A_i}, resample Y_i indep.ly

Note that $|\text{supp}(X_{A_i}, Y_i^{(1)}, \cdots, Y_i^{(k)})| \leq \sum_{y \in \mathbb{Z}^{A_i}} f_i(y)^k$ ▯

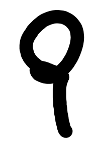
indep sets / colorings / hom in d-reg graphs

Q Fix G, which H maximizes $\hom(H,G)^{\frac{1}{|V(H)|}}$?

Remark This problem is interesting b/c when $G =$ 

$$\hom(H,G) = \#\ \text{indep. sets in } H \quad i(H)$$

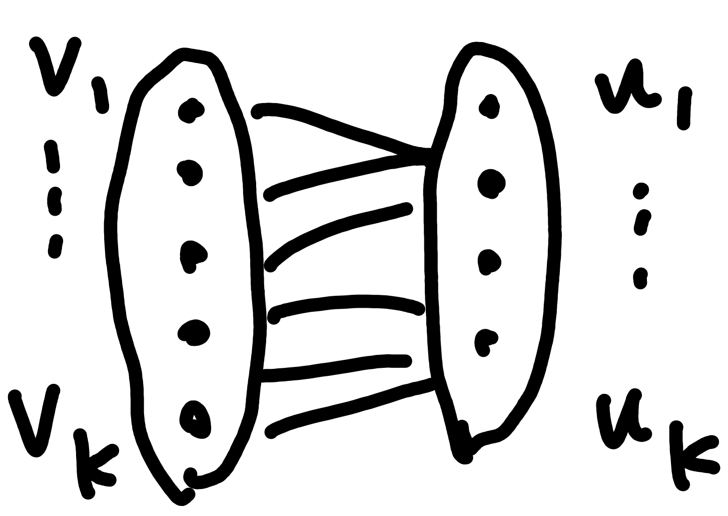
indep. sets ↑ set contains no edge

Thm (Kahn, $G =$  ; Galvin–Tetali)

H : n vtx, d-regular bipartite

$$\hom(H,G)^{\frac{1}{n}} \le \hom(K_{d,d}, G)^{\frac{1}{2d}}$$

Pf H:



Pick $f \in \mathrm{Hom}(H,G)$ u.a.r.

$X_i = f(v_i)$

$Y_i = f(u_i)$

$$\log_2 \hom(H,G) = H(X_1, \cdots, X_k, Y_1, \cdots, Y_k)$$

Take $A_i = \{ j \mid v_j \in N(u_i) \}$ covers $[r]$ d times.
$i = 1, \cdots, k$

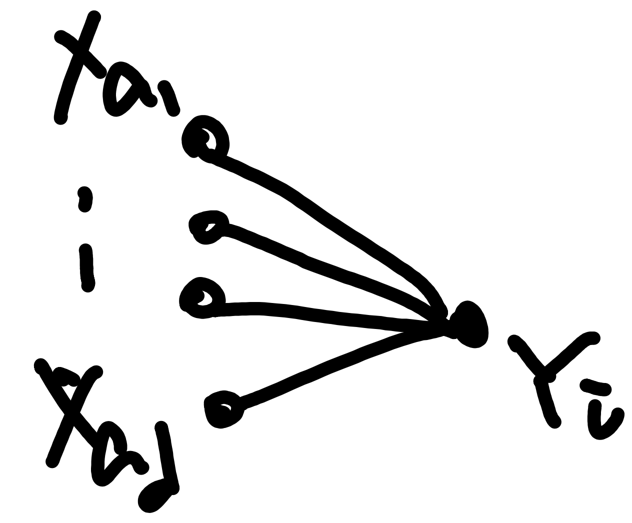
$$d\,H(X_1, \cdots, X_k, Y_1, \cdots, Y_k)$$

$$= d\,H(X_1, \cdots, X_k) + d\,H(Y_1, \cdots, Y_k \mid X_1, \cdots, X_k)$$

$$\text{with } d\,H(X_1, \cdots, X_k) \le \sum H(X_{A_i}) \quad \text{and} \quad d\,H(Y_1, \cdots, Y_k \mid X_1, \cdots, X_k) \le d \cdot \sum H(Y_i \mid X_1, \cdots, X_k) \le d \sum H(Y_i \mid X_{A_i})$$

Thus, $d H(X_1, \dots, X_k, Y_1, \dots, Y_k)$

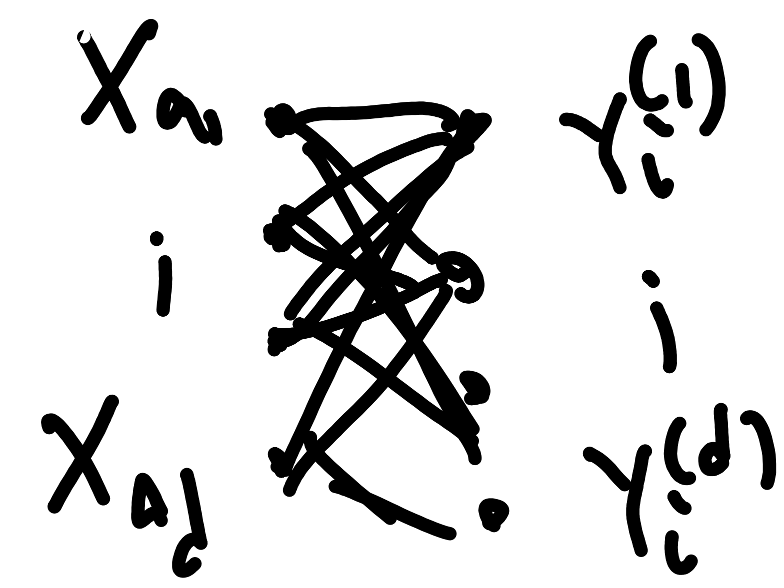
$$\le \sum_{i=1}^{k} H(X_{A_i}) + d H(Y_i \mid X_{A_i})$$



$A_i = \{a_1, \dots, a_d\}$

$$= \sum_{i=1}^{k} H(X_{A_i}, Y_i^{(1)}, \dots, Y_i^{(d)})$$

where $Y_i^{(1)}, \dots, Y_i^{(d)}$ are resampled copies of Y_i conditionally independent given X_{A_i}



$$\le k \log_2 \hom(K_{d,d}, G) \quad \square$$

Remark

Thm (Zhao)

H: d-reg, $i(H)^{\frac{1}{n}} \le i(K_{dd})^{\frac{1}{2d}}$ (n vtx)

Thm (Sah, Sawhney, Stoner, Zhao)

H: d-reg, Δ-free, $\hom(H,G) \le \prod_{uv \in E(H)} \hom(K_{d_u, d_v}, G)^{\frac{1}{d_u d_v}}$

Thm (SSSZ)

H: d-reg $\quad$ " $\quad$ holds for $G = K_q^{\circ \ell}$: K_q w/ ℓ self loops

Conj (SSSZ) Same holds for G: weighted, antiferomagnetic (one possitive eigen value)

Conj (Balogh, Bollobás, Narayanan)

H: 3-unif d-reg, (n vtx) then $i(H)^{\frac{1}{n}} \le i(\quad)^{\frac{1}{3d}}$

Triangle-intersecting graph family

Q $\mathcal{F}$ family of subsets of $[n]$ s.t.

$|A \cap B| \geq 1 \quad \forall A, B \in \mathcal{F}$, max $|\mathcal{F}|$?

Easy! $\boxed{2^{n-1}}$

Q $\mathcal{F}$: family of simple graphs on $V = [n]$ s.t.

$A \cap B$ contains a triangle. $\forall A, B \in \mathcal{F}$

max $|\mathcal{F}|$?

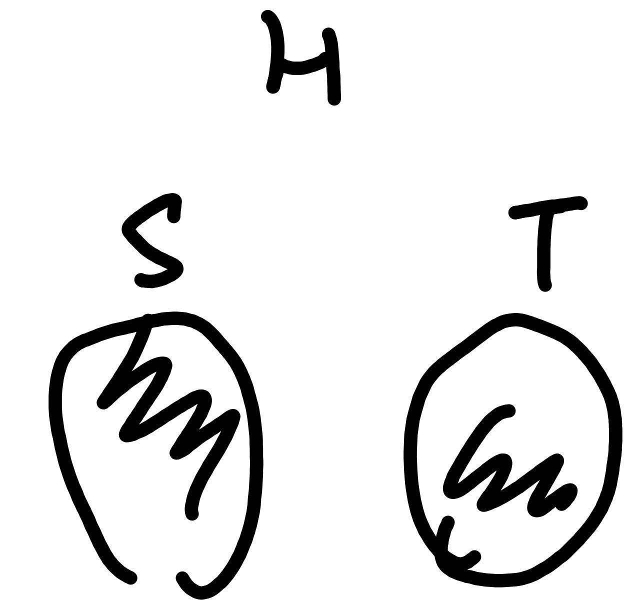
Construction

$\#\{$All graphs containing $12, 23, 13\} = \frac{1}{8} 2^{\binom{n}{2}}$

Upper Bound

Thm (Chung, Graham, Frankl, Shearer)

$$|\mathcal{F}| \leq \frac{1}{4} 2^{\binom{n}{2}}$$



pf (n even) If $H = K_S \cup K_T \quad S \dot\cup T = [n]$

then $A \cap B \cap H$ has at least one edge. $\forall A, B \in \mathcal{F}$

Let $H_1, \cdots, H_r$ be all such graphs. , $r = \binom{n}{\frac{n}{2}}/2$

then generalized Hölder $\Rightarrow |\mathcal{F}|^k \leq \prod_{i=1}^{r} |\mathcal{F}|_{H_i}|$

$\mathcal{F}|_{H_i} = \{A \cap H_i \mid A \in \mathcal{F}\}$, $k = \binom{n-2}{\frac{n}{2}}$

$$\left|\mathcal{F}|_{H_i}\right| \leq 2^{E(H_i)-1} \quad \text{by the previous } \underline{Q}$$

$$= 2^{2\binom{n/2}{2}-1}$$

$$\Rightarrow |\mathcal{F}| \leq 2^{\left(2\binom{n/2}{2}-1\right)\cdot \frac{r}{k}}, \quad \frac{r}{k} = \frac{n(n-1)}{2\frac{n}{2}\left(\frac{n}{2}-1\right)} = \frac{(n-1)}{\left(\frac{n}{2}-1\right)}$$

$$= 2^{\left(2\binom{n/2}{2}-1\right)\cdot \frac{n-1}{\frac{n}{2}-1}}$$

$$\leq 2^{\binom{n}{2}-2}$$

<u>Thm</u> (Ellis, Filmus, Friedgut)

$$|\mathcal{F}| \leq \frac{1}{8} 2^{\binom{n}{2}}$$

<u>Kruskal-Katona type problems</u>

<u>Q</u> max # Δ in a m-edge graph?

<u>Thm</u> (Kruskal; Katona)

If $\# \Delta = \binom{a}{3} + \binom{b}{2}$ w/ $a > b \geq 2$

then $m \geq \binom{a}{2} + \binom{b}{1}$

<u>Thm</u> (Lovász ver)

If $\# \Delta = \binom{x}{3}$ $x \geq 3$

then $m \geq \binom{x}{2}$

Easy: If $\# \Delta = \frac{x^3}{3!}$

then $m \geq \frac{x^2}{2!}$

Pf Let (X_1, X_2, X_3) be a u.a.r. Δ in G

$$\log_2(3!\,\#\Delta) = H(X_1, X_2, X_3)$$

$$2H(X_1, X_2, X_3) \leq 3H(X_1 X_2)$$

$$H(X_1, X_2) \leq \log_2(2!\,m)$$

$$\Rightarrow 2\log_2(3!\,\#\Delta) = 6\log_2 x \leq 3\log_2(2!\,m)$$

□

Thm If $|E(G)| = m$, then $\hom(C_5, G) = O(m^{\frac{5}{2}})$

Pf Let $(X_1, \cdots, X_5)$ be a u.a.r. C_5 in G

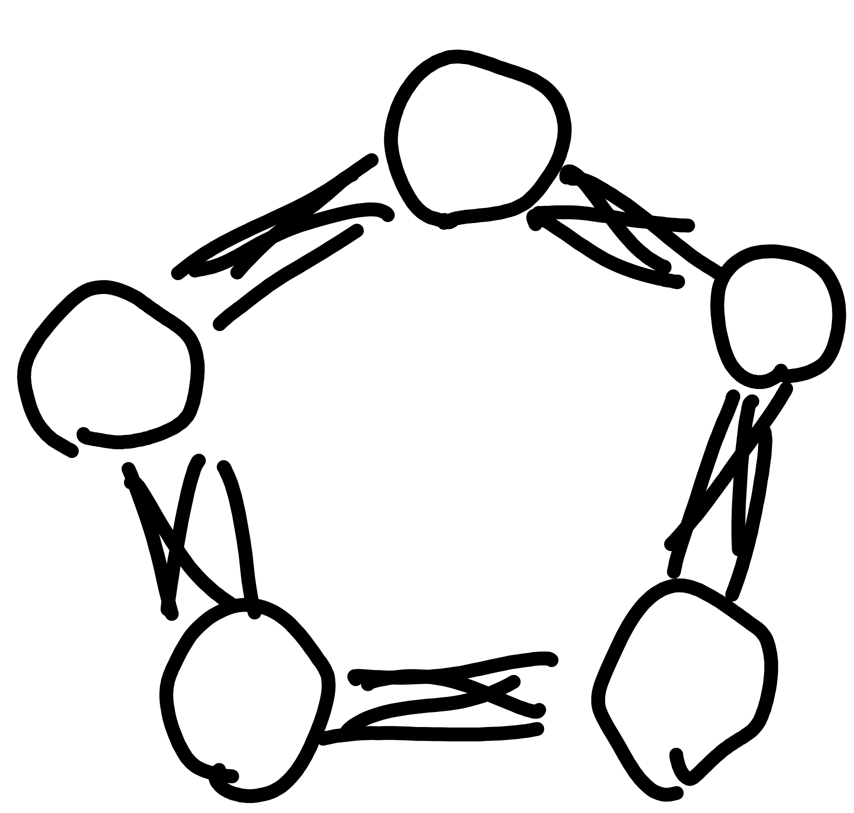
$$\log_2 \hom(C_5, G) = H(X_1, \cdots, X_5)$$

$$2H(X_1, \cdots, X_5) \leq \sum_{i=1}^{5} H(X_i, X_{i+1}) \leq \sum_{i=1}^{5} \log_2(2m)$$

$$\Rightarrow \log_2 \hom(C_5, G) \leq \frac{5}{2}\log_2(2m)$$

□

Tight:



has $\Omega(m^{\frac{5}{2}})$ C_5's