Remarks: (i) (Motivation behind $2\sqrt{d-1}$). The infinite d-regular tree T_d is universal cover for all d-regular ~~graphs~~ graphs, and it has spectral radius $2\sqrt{d-1}$. (Also, motivates the proof)

(ii) Sometimes stated as follows: let d be a constant, $n \to \infty$. Then $\lambda_2 \geq 2\sqrt{d-1} - o_n(1)$. In this formulation, we really require d to be a constant.

(iii) Friedman '08: Let $\varepsilon > 0$, let d be a constant. Let $n \to \infty$ and let G be a d-regular graph chosen uniformly at random on n vertices. Then $\lambda_2 \leq 2\sqrt{d-1} + \varepsilon$ whp.

(iv) When $D \geq 4$, the statement implies that $\lambda_2 = \Omega(\sqrt{d})$. In contrast, when $D \leq 3$, the bound is useless. This can already happen around $d \sim n^{1/3}$.

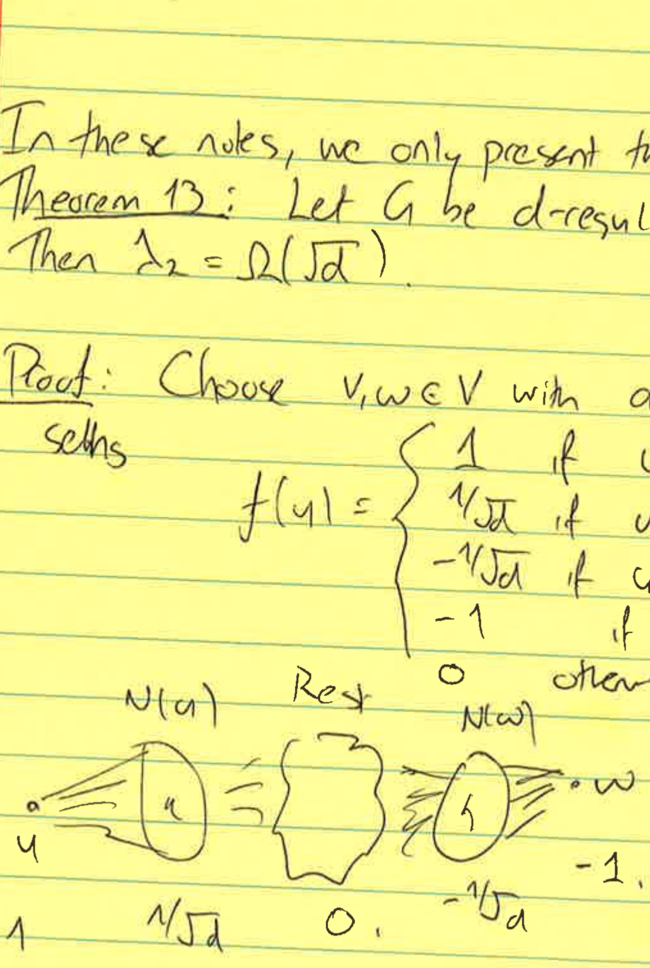
(v) Since $\lambda_2(K_{n/2, n/2}) = 0$, some condition on d is needed.

In these notes, we only present the proof of the weaker version

Theorem 13: Let G be d-regular graph with diameter $D \geq 4$. Then $\lambda_2 = \Omega(\sqrt{d})$.

Proof: Choose $v, w \in V$ with $d(v,w) \geq 4$. Define $f \in \mathbb{R}^{V(G)}$ by setting

$$f(u) = \begin{cases} 1 & \text{if } u = v \\ 1/\sqrt{d} & \text{if } uv \in E \\ -1/\sqrt{d} & \text{if } uw \in E \\ -1 & \text{if } u = w. \\ 0 & \text{otherwise} \end{cases}$$



Then $\langle f, \mathbb{1} \rangle = 0$ as $|N(u)| = |N(w)| = d$, and $\|f\|_2^2 = 1 + d \cdot 1/d + d \cdot 1/d + 1 = 4$.
$\Rightarrow \|f\|_2 = 2$.

Recall that $\lambda_2 = \sup_{\substack{\langle w, \mathbb{1}\rangle = 0 \\ w \neq 0}} \frac{\langle Aw, w\rangle}{\langle w, w\rangle} \geq \frac{\langle Af, f\rangle}{\langle f, f\rangle}$.

Note that (i) For $u = v$, we have $(Af)(v) = \sum_{u' \in N(v)} f(u') = d \cdot 1/\sqrt{d} = \sqrt{d}$.

(ii) $u = w$, we similarly have $(Af)(w) = -\sqrt{d}$.

(iii) For $u \in N(v)$, we have

$$(Af)(u) = \underbrace{\left(\sum_{u' \in N(u) \cap N(v)} f(u')\right)}_{\geq 0 \text{ as all terms are } 1/\sqrt{d} \text{ (but the sum might be empty)}} + \underbrace{(f(v))}_{=1} + \overbrace{\sum_{u' \in N(u) \setminus N(v)} f(u')}^{=0}$$

as $N(u) \subseteq N^{\leq 2}[v] \subseteq N(w)^c$ as $d(v,w) \geq 4$.

≥ 1.

(iv) For all $u \in N(w)$, we have $(Af)(u) \leq -1$ similarly.

Thus $\langle f, Af\rangle \geq 1 \cdot \sqrt{d} + d \cdot 1/\sqrt{d} \cdot 1 + d \cdot (-1/\sqrt{d}) \cdot (-1) + (-1)(\sqrt{-d})$
$= 4\sqrt{d}$.

Hence $\lambda_2 \geq 4\sqrt{d}/4 = \sqrt{d}$ $\square$.

When proving the bound $\lambda_2 \geq 2\sqrt{d-1} - \frac{2\sqrt{d-1}+1}{\lfloor D/2 \rfloor}$, we fix edges e_1, e_2 far apart, define sets $V_i = N^i(e_1)$ & $W_i = N^i(e_2)$, and define vector f that is constant on each V_i & W_j (>0 on V_i, <0 on W_j).

<u>Exercise 14</u>: Let d be constant, $\varepsilon > 0$. Prove that $\exists c > 0$ so that whenever G is d-regular & n suff. large, then G contains $\geq cn$ eigenvalues $> 2\sqrt{d-1} - \varepsilon$.

<u>Discrepancy:</u> Given $U \subseteq V(G)$, define $disc(U) = e(U) - p\binom{|U|}{2}$.
Note that $p\binom{|U|}{2}$ is the expected number of edges in a random subset U' of size $|U|$ in G, so $disc(U)$ measures the deviation of $e(U)$ from its expectation. We define discrepancy, positive discrepancy and negative discrepancy by setting

$$disc(G) = \max_{U \subseteq V(G)} |disc(U)|.$$

$$disc^+(G) = \max_{U \subseteq V(G)} disc(U).$$

$$disc^-(G) = \max_{U \subseteq V(G)} -disc(U).$$

The study of discrepancy dates back to the work of Erdős and Spencer in '71 who proved that $disc(G) = \Omega(n^{3/2})$. when $p = 1/2$. This was later extended to all values of p.

<u>Theorem 15:</u> (Erdős, Goldberg, Pach, Spencer '88). Suppose that $1/n \le p \le 1 - 1/n$. Then $disc(G) = \Omega(\sqrt{p(1-p)n^3})$.

<u>Remarks</u> (i) The range for p is optimal (check this!).
(ii) The bound is tight (up to multiplicative constants) for $G(n,p)$ (check this!)
(iii) In particular, it follows that at least one of $disc^-$ or $disc^+$ must be large. But the result doesn't tell anything on which one of them is going to be large, nor how large must the smaller one be.

<u>Exercise 16:</u> By applying ~~Turán~~ arguments similar to the proof of Turán's theorem (iteratively remove vertex of max degree), prove that $disc^+(G), disc^-(G) = \Omega(n)$. whenever $1/n \le p \le 1 - 1/n$.

Note that $disc^+(T_r(n)) = \Theta(n)$ and $disc^-((T_r(n))^c) = \Theta(n)$, so the lower bound cannot be improved (globally).

We now discuss some similarities & differences of $disc^{\pm}(G)$, $sp(G)$, λ_2, λ_n & $bw(G)$.
Here, $bw(G)$, the bisection width of G, is defined as

$$bw(G) = \min_{\substack{A \cup B = V \\ A \cap B = \phi \\ \left||A|-|B|\right| \leq 1}} e(A,B) \qquad (\text{"Min-Cut" among equipartitions}),$$

$$dfc(G) = m/2 - e(A,B) \qquad (\text{deficit of } G).$$

and $disc(U,W) := e(U,W) - p|U||W|$.

Exercise 19: (i) Prove that $disc(U) + disc(U,W) + disc(W) = disc(U \cup W)$.

(ii) Suppose that $1/n \leq p \leq 1 - 1/n$. Prove that
$disc^{-}(G) = \Omega(sp(G))$ & $disc^{+}(G) = \Omega(dfc(G))$.

(iii) Prove that the converse statements need not hold.

(iv) Suppose that G is regular, & $1 \leq d \leq n-2$.
Prove that $sp(G) = \Theta(disc^{-}(G))$ & $dfc(G) = \Theta(disc^{+}(G))$.

(v) Let G be d-regular. Prove that $\forall U \subseteq V(G)$,

$$\frac{\lambda_n}{2}|U| - p|U| \leq disc(U) \leq \frac{\lambda_2}{2}|U| + p|U|.$$

Corollary 20: $(\lambda_2+1)|\lambda_n| = \Omega(d)$ whenever G is d-regular & $d \leq (1-\varepsilon)n$.

Proof: Follows from Exercise 19(v) + Bollobás-Scott $\square$.

Exercise 21: Prove that Bollobás-Scott is tight (up to multiplicative constants) for strongly regular graphs.