Given a graph G, we write
- m for the number of edges in G
- n for the number of vertices in G.
- $p = m/\binom{n}{2}$ for the edge-density.
- $d = p(n-1)$ for the average degree.

A cut is a bipartition $V = A \cup B$. The size of a cut $A \cup B$ is $e(A,B)$, the number of edges with exactly one endpoint in both A and B. Max-Cut is defined as the largest size of a cut, ie. $\max_{A \cup B = V} e(A,B)$. We write $mc(G)$ for the max-cut of G.

Exercise 1: Prove that $mc(G) \geq m/2$ (probabilistic method).

Define the surplus of G by setting $sp(G) = mc(G) - m/2$. Thus $sp(G)$ measures how much $mc(G)$ exceeds the expected size of a cut. It is natural to ask: given some properties of G, how large must $sp(G)$ be (in terms of $m, n,$ etc.). First such result was obtained by Edwards in '73.

Theorem 2 (Edwards, '73). $sp(G) \geq \frac{\sqrt{8m+1}-1}{8}$, and the equality holds if $G = K_{2t+1}$ for some t.

From now on, we do not care about exact bounds, but rather on their asymptotic growth rate. It is not too hard to verify that if G is "close" to "a union of constant number of cliques", then $sp(G) = \Theta(\sqrt{m})$, thus asymptotically matching the bound of Edwards. (Here, we are intentionally a bit vague).
It is natural to ask under what conditions can bounds on $sp(G)$ be improved.

One such direction was established by Erdős and Lovasz in '70s. They started the study of $sp(G)$ for H-free graphs (here H is a fixed graph).

This additional restriction certainly ensures that G is far away from the constructions mentioned above.

<u>Conjecture 3</u> (Alon, Bollobás, Krivelevich, Sudakov '03).
Let H be a given graph. Then there exists $\varepsilon_H > 0$ so that $sp(G) = \Omega(m^{3/4+\varepsilon_H})$ whenever G is H-free.

They proved existence of $\varepsilon_H' > 0$ so that $sp(G) = \Omega(m^{1/2+\varepsilon_H'})$.

Progress towards the conjecture and results of similar flavour.
(i) Many results of the form: $\forall t\ \exists \varepsilon_t' > 0$ so that G K_t-free, then $sp(G) = \Omega(m^{1/2+\varepsilon_t'})$. Pre-2025, one always had $\varepsilon_t' \to 0$ as $t \to \infty$.
(ii) Zhang (25+) $\exists \varepsilon' > 0$ so that $sp(G) = \Omega(m^{1/2+\varepsilon'})$ whenever G is K_t-free (uniform ε' that beats the $1/2$ barrier).
(iii) Jin, Milojević, Tomon, Zhang (25+): $\forall \delta > 0\ \exists \varepsilon > 0$ s.t. if G doesn't contain a clique of size $m^{1/2-\delta}$, then $sp(G) = \Omega(m^{1/2+\varepsilon})$.
(iv) Stability on Edwards' bound at the very other end of regime: Balla, Hambardzumyan, Tomon (26): If $sp(G) = \alpha\sqrt{m}$, then G contains a clique of size $f(\alpha)\sqrt{m}$. (their argument gives $f(\alpha) = 2^{-O(\alpha^9)}$).

Last three are too technical to be covered, but we'll give a proof of the first one. Later, we also prove that $sp(G) = \Omega(m^{4/5})$ whenever G is triangle-free, and $4/5$ is tight.

The following exercises are used later as well, and certainly imply the asymptotic variant of the Edwards bound $sp(G) = \Omega(\sqrt{m})$.

Exercise 4: Let G be a graph of chromatic number χ, and with m edges. Then $sp(G) = \Omega(m/\chi)$. In particular,

$sp(G) = \Omega(m/n)$.

Exercise 5: Let $v_1, \ldots, v_n$ be an ordering of the vertices in G. Let $I = \{j : v_j$ has odd number of neighbors in $\{v_1, \ldots, v_{j-1}\}\}$.
Prove that there exists constants $C_1, C_2 > 0$ so that

(i) $sp(G) \geq C_1 |I|$.

(ii) If G doesn't contain isolated vertices, then $\exists$ ordering for which $|I| \geq C_2 n$. (and thus $sp(G) = \Omega(n)$ whenever G doesn't contain isolated vertices).

(iii) Deduce that $sp(G) = \Omega(\sqrt{m})$.

Follows approach by Carlson et.al. →

We now prove that $\exists \varepsilon_t' > 0$ so that $sp(G) = \Omega(m^{1/2 + \varepsilon_t})$ whenever G is K_t-free.

Lemma 6: Suppose that G is K_t-free. Then $\chi(G) \leq 4n^{\frac{t-2}{t-1}}$.

Proof: Let $f(n) = 4n^{\frac{t-2}{t-1}}$. We prove the statement by induction. If $n \leq 4^{t-1}$, the statement follows from $\chi(G) \leq |G|$.

Now suppose that $n > 4^{t-1}$. By Ramsey's thm $\left(R(s,t) \leq \binom{s+t-2}{t-1} \leq s^{t-1}\right)$, if G is K_t-free then $\alpha(G) \geq \lfloor n^{1/(t-1)} \rfloor$.
Let $s := \lfloor n^{1/(t-1)} \rfloor$ and pick $I \subseteq G$, $|I| = s$, I independent set.

As $\chi(G) \leq 1 + \chi(G[V \setminus I]) \leq 1 + f(n-s)$ by induction, it suffices to show that $f(n) \geq f(n-s) + 1$ for all $n \geq 4^{t-1}$.
This is indeed true, but a bit painful calculation. We leave it as an exercise. □

Theorem 7: Suppose G is K_t-free, $(t \geq 3)$. Then $sp(G) = \Omega(m^{1/2 + \varepsilon_t})$ for $\varepsilon_t = \frac{1}{4t-6}$.

Proof: Exercises 4&5 + Lemma 6 $\Rightarrow$ (assuming G doesn't contain isolated vertices, which we can always assume)
$sp(G) = \Omega(n)$.
and $sp(G) = \Omega\left(\frac{m}{n^{\frac{t-2}{t-1}}}\right)$.

Thus $sp(G)^{\alpha} = \Omega(m)$, where $\alpha = 1 + \frac{t-2}{t-1} = \frac{2t-3}{t-1}$.

As $sp(G) = \Omega(m^{1/\alpha})$ and $1/\alpha = \frac{t-1}{2t-3} = \frac{1}{2} + \frac{1}{4t-6} = \frac{1}{2} + \varepsilon_t$

the result follows $\square$.

Necessary background on algebraic graph theory

Given G, we write A for the adjacency matrix of G, ie.

$$A_{ij} = \begin{cases} 1 & \text{if } ij \in E \\ 0 & \text{otherwise (including } i=j). \end{cases}$$

Since A is real symmetric matrix, it has a set of n orthogonal unit eigenvectors $v_1, \dots, v_n$ associated with real eigenvalues $\lambda_1 \geq \dots \lambda_n$. The following facts are well-known, and verifying them is left as an exercise.

Exercise 8: Prove that

(i) $d \leq \lambda_1 \leq \Delta$, where Δ = max degree in G.

(ii) G is Δ-regular $\Leftrightarrow \lambda_1 = \Delta \Leftrightarrow$ we may choose $v_1 = \frac{1}{\sqrt{n}}(1, \dots, 1)$ (ie. $(1, \dots, 1) \in E_{\lambda_1}$, the λ_1-eigenspace).

(iii). G is Δ-regular and connected $\Leftrightarrow$ $E_{\lambda_1} = \mathrm{span}\{(1, \dots, 1)\}$.

(iv) We have $\sum_{i=1}^{n} \lambda_i^k$ = # labeled closed walks of length k in G.

In particular, $\sum \lambda_i = 0$, $\sum \lambda_i^2 = nd = 2e(G)$, $\sum \lambda_i^3 = 6$ # triangles.

(v) If $d \leq (1-\varepsilon)n$, then $\max\{\lambda_2, |\lambda_n|\} = \Omega(\sqrt{d})$.

An important topic that we will not cover, but which shares similar flavor is expansion

Theorem 9 (Expander mixing lemma). Let G be a d-regular graph, and write $\lambda = \max\{\lambda_2, |\lambda_n|\}$. Then for all $S, T \subseteq V(G)$ (not necessarily disjoint), we have

$$\left| e(S,T) - \frac{d}{n}|S||T| \right| \leq \lambda \sqrt{|S||T|\left(1-\frac{|S|}{n}\right)\left(1-\frac{|T|}{n}\right)}.$$

The term $d/n\, |S||T|$ should be interpreted as the "expected value of $e(S,T)$". Thus, the expander mixing lemma states that the deviation of $e(S,T)$ from its expectation is upper bounded by λ.

Important variants and consequences:

Lemma 10 (Variant for $T = S^c$). Let G be d-regular graph. Then

$$-\frac{\lambda_2}{n}|S||S^c| \leq e(S,S^c) - \frac{d}{n}|S||S^c| \leq \frac{|\lambda_n|}{n}|S||S^c|.$$

Theorem 11 (Hoffman bound) Let G be d-regular graph. Then

$$\alpha(G) \leq \frac{|\lambda_n|\, n}{d + |\lambda_n|}.$$

Expander mixing lemma implies that λ_2 and λ_n play important role in bounding the expansion properties of G, and also play important role in bounding maxcut (Lemma 10), and positive and negative discrepancies (we will discuss these later). Thus, it is natural to ask how small can λ_2 be.

Theorem 12 (Alon-Boppana bound, '91). Let G be a d-regular graph of diameter D. Then $\lambda_2 \geq 2\sqrt{d-1} - \dfrac{2\sqrt{d-1}-1}{\lfloor D/2 \rfloor}$.