Remarks: (i) Tight when $d \le n^{3/4}$.
(ii) Not interesting when $d = \Theta(n)$.

Corollary 36 Let G be a d-regular graph. Then
$$\lambda_2 \ge \begin{cases} \sqrt{d} & \text{if } d \le n^{2/3} \\ n/d & \text{if } n^{2/3} \le d \le an \text{ for a small enough constant.} \end{cases}$$

Proof: Follows immediately from Theorem 35 $\square$.

We now prove improved lower bound for λ_2 when d is large. This regime was already studied by Balla (2C), motivated by equiangular lines.

Theorem 37 (Balla '2G; see also Ihringer '23, R., Sudakov, Tomon '2G, Balla, R., Sudakov, Tomon '23+).
Given d-regular graph on G with $d \le (1/2 - \varepsilon) n$, then $\lambda_2 = \Omega(|\lambda_n|^{1/2})$. In particular, $\lambda_2 = \Omega(d^{1/3})$.

We present the proof of BRST. Given matrices A, B, the Hadamard product $A \circ B$ is given by $(A \circ B)_{ij} = A_{ij} B_{ij}$. If A, B are positive semidefinite, then so is $A \circ B$ (exercise!).

Proof. Define $M = \lambda_2 I - A + \alpha J$, $\alpha = \frac{1}{n}(d - \lambda_2)$. Then $M\mathbb{1} = 0$, $Mv_i = (\lambda_2 - \lambda_i)$ $\forall i \ge 2 \Rightarrow M$ positive semidefinite.
Then $X := M \circ M = \lambda_2^2 I + A + 2\alpha\lambda_2 I - 2\alpha A + \alpha^2 J$.

$\Rightarrow$ $Xv_n = \mu_n v_n$, where $\mu_n = (1-2\alpha)\lambda_n + \lambda_2(\lambda_2 + 2\alpha)$
Since X is positive semidefinite, we have $\lambda_2(\lambda_2 + 2\alpha) \ge (1-2\alpha)|\lambda_n|$ as $\mu_n \ge 0$. Since $\lambda_2 > 0$, we have
$\lambda_2^2 = \Omega(\lambda_2(\lambda_2 + \alpha)) \ge \Omega((1-2\alpha)|\lambda_n|)$.
Also, $1 - 2\alpha = 1 - \frac{2(d - \lambda_2)}{n} \ge 1 - \frac{2d}{n} \ge 2\varepsilon$

$\Rightarrow$ $\lambda_2 = \Omega(|\varepsilon\lambda_n|^{1/2}) = \Omega_\varepsilon(|\lambda_n|^{1/2})$. By using $(\lambda_2 + 1)|\lambda_n| = \Omega(d)$, then $\lambda_2 = \Omega(d^{1/3})$ $\square$.

Recently, the regime when $d \geq (\frac{1}{2} - \varepsilon)n$ was addressed by Zhang:

<u>Theorem 38</u> (Zhang). Let $\varepsilon > 0$. Then $\exists \varepsilon' > 0$ s.t. whenever $|G \Delta T_r(n)| \geq \varepsilon' n^2$, we have $\lambda_2 \geq n^{1/4 - \varepsilon}$.

The exponent $1/4$ is the best possible. See also Jin, Milojević, Tomon, Zhang who proved analogous results under the ~~[illegible]~~ weaker condition $|G \Delta T_r(n)| \geq \alpha \cdot n^{\delta}$ for given δ.

<u>Lower bounds for $\mathrm{disc}^+(G)$ when $d \leq (\frac{1}{2} - \varepsilon)n$.</u>

<u>Theorem 39</u> (Graph Grothendieck's inequality; Alon, Makarychev, Makarychev, Naor '06). Let A be arbitrary matrix. Then $\exists K > 0$ so that

$$\max_{x_1, \dots, x_n \in \{-1,1\}} \sum_{i \neq j} A_{ij} x_i x_j \geq \sup_{v_1, \dots, v_n \in S^{n-1}} \frac{K}{\log n} \sum_{i \neq j} A_{ij} \langle v_i, v_j \rangle$$

Let X be a positive semidefinite matrix with $X_{ii} \leq 1$.
Let $\mathrm{disc}_*(X) = \langle X, \underbrace{A}_{\text{Adjacency matrix of } G} - pJ \rangle_F$, where

$\langle A, B \rangle_F = \mathrm{tr}(A^T B)$ is the Frobenius inner product.

Let $\mathrm{disc}_*^+(G) = \max_X \langle X, A - pJ \rangle_F$ where max is taken over all positive semidefinite X so that $X_{ii} \leq 1$.

& $\mathrm{disc}_*^-(G) = \max_X \langle -X, A - pJ \rangle_F$.

Applying Grothendieck's inequality to $A - pJ$, we conclude that $\operatorname{disc}^+(G) \geq \frac{K}{\log n} \sup_{v_1 \to v_n \in S^{n-1}} \sum_{i \neq j} (A-pJ)_{ij} \langle v_i, v_j \rangle$.

$$\geq \frac{K}{\log n} \sup_{v_1 \to v_n \in S^{n-1}} \left(\sum_{i,j} (A-pJ)_{ij} \langle v_i, v_j \rangle \right)$$

as the added terms $-p \sum_i \langle v_i, v_i \rangle$ are all negative.

Define $X_{ij} = \langle v_i, v_j \rangle$. Then X is clearly positive semidefinite (it is a Gram matrix), and $X_{ii} \leq 1$

Conversely, given positive semidefinite matrix X, $\exists\, v_1 \to v_n$ such that $X_{ij} = \langle v_i, v_j \rangle$ (think of LU factorization).

Thus $\operatorname{disc}^+(G) \geq \frac{K}{\log n} \operatorname{disc}^+_*(G)$.

and we trivially have $\operatorname{disc}^+_*(G) + d \geq \operatorname{disc}^+(G)$. (1).
(+d because of $p \sum_{i=1}^n \langle v_i, v_i \rangle$ term).

Similarly, we have $\operatorname{disc}^-_*(G) + d \geq \operatorname{disc}^-(G) \geq \frac{\operatorname{disc}^-_*(G) \cdot K}{\log n}$. (2)

<u>Lemma 40</u> Suppose that $\operatorname{disc}^+_*(G), \operatorname{disc}^-_*(G) \geq \Omega(n)$.
Then $\operatorname{disc}^+_*(G) \operatorname{disc}^-_*(G) = \Omega(dn^2)$.

<u>Proof</u>: Follows from Bollobás - Scott. Note that is kind of needed due to +d terms in the inequalities (1) & (2)

To prove lower bounds for disc^+, we focus on lower bounding disc_*^+.

<u>Lemma 41</u>: $\mathrm{disc}_*^+(G) \geq \begin{cases} \sum_{\lambda_i > 0, i \geq 2} \lambda_i \\ \sum_{\lambda_i > 0, i \geq 2} \lambda_i^2 / \sqrt{d}. \\ \sum_{\lambda_i > 0, i \geq 2} \lambda_i^3 / d. \end{cases}$

<u>Proof</u>: For the first one, pick $X = \sum_{i=2}^{k} v_i v_i^T$, k chosen so that $\lambda_k \geq 0$, $\lambda_{k+1} < 0$.

Now, as $\sum_{i=1}^{n} v_i v_i^T = I$ & each $v_i v_i^T$ is positive semidefinite,

X is pos. semidefinite & $X_{ii} \leq (I)_{ii} = 1 \ \forall i$.

Thus $\mathrm{disc}_*^+(G) \geq \mathrm{disc}(X) = \langle \sum_{i=2}^{k} v_i v_i^T, \sum_{j=2}^{n} \lambda_j v_j v_j^T \rangle$

$= \sum_{i=2}^{k} \sum_{j=2}^{n} \lambda_j \langle v_i, v_j \rangle^2 = \sum_{i=2}^{k} \lambda_i$, proving the first bound.

The third follows by taking $X = \frac{1}{d} \sum_{i=2}^{k} \lambda_i^2 v_i v_i^T$.

Clearly, X is pos. semidefinite. Also $1/d\,(A^2) = \frac{1}{d} \sum_{i=1}^{n} \lambda_i^2 v_i v_i^T$

so $1/d\, A^2 - X$ is pos. semidefinite.

$\Rightarrow \quad X_{ii} \leq (1/d\, A^2)_{ii} = 1/d \cdot d = 1 \ \forall i$.

The second follows from first & third & CS. $\square$.

<u>Corollary 42:</u> $\mathrm{disc}_*^+(G) \geq \lambda_2^3/d$.

<u>Theorem 43</u> (R., Sudakov, Tomon '26). Let G be a graph with $d \leq (1/2 - \varepsilon)n$. Then $\mathrm{disc}^+(G) = \Omega\left(\frac{nd^{1/4}}{\log n}\right)$.

<u>Theorem 44:</u> (Zhang 25+, see also JMTZ '26).
If $|G \Delta T_r(n)| \geq \varepsilon n^2$ then $\mathrm{disc}^+(G) = \Omega(n^{1+\delta})$.

Here, we prove only weaker variant of Theorem 43 with $nd^{1/4}$ replaced by a slower growing function, which is still superlinear. We also assume that G is regular.

A key tool in the proof is the following "positive" relation between $\mathrm{disc}_*^+(G)$ & $\mathrm{disc}_*^-(G)$.

<u>Lemma 45:</u> If $p \leq 1/2 - \varepsilon$, then
$$\mathrm{disc}_*^+(G) \geq -2d + \frac{(1-2p)}{\lambda_2}\mathrm{disc}_*^-(G).$$
(Here, one should think $-2d$ as a lower-order error term).

<u>Theorem 46</u> Suppose that $p \leq 1/2 - \varepsilon$. Then $\mathrm{disc}_*^+(G) \geq \Omega\left(d^{2/7} n^{6/7}\right)$.

<u>Proof:</u> Multiply together • bound of Lemma 40 to power 3.
• bound of Corollary 42 & • bound of Lemma 45 to power 3.

Thus $\mathrm{disc}_*^+(G)^7 \cdot \mathrm{disc}_*^-(G)^3 \geq \Omega\left(d^{\cancel{3/2}\,3} \cdot n^6 \cdot \lambda_2^3/d \cdot 1/\lambda_2^3 \cdot \mathrm{disc}_*^-(G)^3\right)$.

$$\Rightarrow \mathrm{disc}_*^+(G) = \Omega\left(d^{2/7} n^{6/7}\right). \square$$

Corollary 47: $\mathrm{disc}^+(G) = \Omega\left(\frac{d^{2/7} n^{6/7}}{\log n}\right)$ whenever $d \le (1/2 - \varepsilon)n$.

In particular, $\mathrm{disc}^+(G) = \Omega\left(\frac{n^{1+1/7}}{\log n}\right)$ when $d = \alpha n$

for $\alpha \le 1/2 - \varepsilon$.

Proof of Lemma 45: Pick X so that $\mathrm{disc}_{\bar{\lambda}}(G) = -\mathrm{disc}(X)$.

Wlog $\langle X, J\rangle_F = 0$, as we can always replace X with $X + \alpha J$.
(Note that $|\alpha| \le 1$, so we may have to scale X down by a factor of $1/2$, but this will only affect the multiplicative constants).
Let $Y = I - 1/\lambda_2 A + \frac{p}{\lambda_2} J$.
Claim: Y is positive semidefinite, and $\mathrm{disc}(X \circ Y) = -p \sum_i X_{ii} - \frac{(1-2p)}{\lambda_2}\mathrm{disc}(X)$.

Proof: $Y v_i = (1 - \lambda_i/\lambda_2) v_i$ & $1 - \lambda_i/\lambda_2 \ge 0 \;\forall i \ge 2$
& $Y v_1 = (1 - d/\lambda_2 + pn/\lambda_2) v_1 = (1 + p/\lambda_2) v_1$ & $1 + p/\lambda_2 \ge 0$.
Thus Y is positive semidefinite. Note that $\langle U \circ V, W\rangle_F = \langle U, V \circ W\rangle_F$

Thus $\mathrm{disc}(X \circ Y) = \langle X \circ Y, A - pJ\rangle_F$.

$$= \langle X \circ I - 1/\lambda_2\, X \circ A + p/\lambda_2\, X, A - pJ\rangle_F.$$

$$= -p \sum_{i=1}^{n} X_{ii} - 1/\lambda_2 \langle X \circ A, A - pJ\rangle_F + p/\lambda_2 \langle X, A - pJ\rangle_F.$$

$$= -p \sum_{i=1}^{n} X_{ii} - 1/\lambda_2 \langle X, A \circ A - p A \circ J\rangle_F + p/\lambda_2\, \mathrm{disc}(X)$$

$$= -p \sum_{i=1}^{n} X_{ii} - 1/\lambda_2 \langle X, (1-p) A\rangle_F + p/\lambda_2\, \mathrm{disc}(X).$$

$$= -p \sum_{i=1}^{n} X_{ii} - \frac{(1-p)}{\lambda_2}\mathrm{disc}(X) + p/\lambda_2\, \mathrm{disc}(X) \quad \text{as } \langle X, J\rangle_F = 0$$

$$= -p \sum_{i=1}^{n} X_{ii} - \frac{(1-2p)}{\lambda_2}\mathrm{disc}(X) \;\square.$$

Since $-\mathrm{disc}(X) = \mathrm{disc}_{\bar{*}}(G)$ and $\rho\left|\sum_{i=1}^{A} X_{ii}\right| \leq \rho n \leq 2d$.

the result follows $\square$.