In the following examples, we for simplicity use the approximation $\arcsin(x) \approx x$. In reality, one should use proper (inequality) estimates, often using different estimates for $x < 0$ & $x \geq 0$. There are plenty of interesting applications where this is enough. There are also a handful of applications where one has to look at higher order terms ($O(x^3)$-terms).

Our first example is proving $sp(G) = \Omega(m^{4/5})$ whenever G is triangle-free. First, where is $4/5$ coming from? A good guess is that pseudorandom triangle-free graphs would be good choice. That is, we have $|\lambda_n|, \lambda_2 \sim \sqrt{d}$.

G triangle-free $\Rightarrow$ $0 = tr(A^3) \geq d^3 + n\lambda_n^3 \Rightarrow d^3 \sim n d^{3/2}$.
$\Rightarrow$ need $d \leq n^{2/3}$ for pseudorandom Δ-free graphs to exist.
For pseudorandom graphs, we'd also have $sp(G) \sim n\sqrt{d} \sim n^{4/3}$.
and $m = \frac{1}{2}nd \sim n^{5/3}$ $\Rightarrow sp(G) \sim m^{4/5}$.
In fact, Alon proved that pseudorandom Δ-free graphs exist with $d \sim n^{2/3}$ for infinitely many n.

<u>Theorem 29 (Alon)</u> $\exists$ infinite sequence $q_k \to \infty$ for which there exists G_k with $n_k \sim q_k^3$, $d_k \sim q_k^2$, $\lambda_2, |\lambda_n| \sim q_k$ which are Δ-free (Thus $sp(G_k) \sim q_k^4 \sim m_k^{4/5}$, $m_k = \frac{1}{2}n_k d_k \sim q_k^5$).

<u>Lemma 30</u> Let G be Δ-free graph with degree sequence $d_1 \to d_n$. Then $sp(G) = \Omega\left(\sum_{i=1}^{n} \sqrt{d_i}\right)$.

Think of G d-regular $\Rightarrow$ $sp(G) = \Omega(n\sqrt{d})$.

Proof: Choose $x_u \in \mathbb{R}^{V(G)}$ with
$$x_u(v) = \begin{cases} 1 & \text{if } u=v \\ 0 & \text{if } uv \notin E \text{ \& } u \neq v \\ -1/\sqrt{d_u} & \text{if } uv \in E. \end{cases}$$

Given $uv \in E$, we have $x_u(s)x_v(s) \neq 0 \Rightarrow s \in N[u] \cap N[v]$

$(N[w] = \{w\} \cup N(w))$. Now, $N(u) \cap N(v) = \phi$ as G is Δ-free.
$\Rightarrow x_u(s)x_v(s) \neq 0 \Rightarrow s \in \{u,v\}$

Thus $\langle x_u, x_v \rangle = -1/\sqrt{d_u} - 1/\sqrt{d_v}$. Also, $\|x_u\|_2^2 = 2 \ \forall u$.

Thus
$$sp(G) \geq -\frac{1}{\Omega} \sum_{uv \in E} \arcsin\left(\frac{\langle x_u, x_v \rangle}{\|x_u\|_2 \|x_v\|_2}\right).$$
$$= \Omega\left(\sum_{uv \in E}\left(1/\sqrt{d_u} + 1/\sqrt{d_v}\right)\right) = \Omega\left(\sum_u \sqrt{d_u}\right) \square$$

"This bound allows us to deal with graphs with "not too many large degrees" (in fact with graphs that are d degenerate for $d \sim m^{2/5}$)". But we need other argument for graphs that are "too dense" (more precisely, for those graphs that contain subgraph with large min degree).

Lemma 31: Suppose G has m edges and is d-degenerate for $d = m^{2/5}$. Then $sp(G) = \Omega(m^{4/5})$.

Proof: Pick $v_1, \dots, v_n$ so that $d_{<i} := |\{j : v_i v_j \in E, j < i\}| < d \ \forall i$.

Thus, $sp(G) = \Omega\left(\sum_{i=1}^{n} \sqrt{d_i}\right) \geq \Omega\left(\sum_{i=1}^{n} \sqrt{d_{<i}}\right) \geq \frac{\Omega\left(\sum_{i=1}^{n} d_{<i}\right)}{\sqrt{d}}$.

$= \frac{\Omega(m)}{\sqrt{d}} = \Omega(m^{4/5})$. as $\sum_{i=1}^{n} d_{<i} = m$. $\square$

~~Exercise~~ **Exercise 32** Suppose $H_1, \ldots, H_t$ are vx disjoint induced subgraphs of G. Then $sp(G) \geq \sum_{i=1}^{t} sp(H_i)$. In particular, $sp(G) \geq sp(H_1)$.

Lemma 33: Suppose that G is not d-degenerate for $d = m^{2/5}$ and $e(G) = m$. Then $sp(G) = \Omega(m^{4/5})$.

Proof: Let H' be induced subgraph with $\delta(H') \geq d$.
Let $R \subseteq V(H')$ be a random subset obtained by choosing $v_1, \ldots, v_r$ from H' uniformly at random (allowing repeats), where $r = \lceil 2|H'|/d \rceil$.
Let H'' denote the induced subgraph of H' so that $v \in H''$ iff $vv_i \in E$ for some $i \in [r]$.

$$\text{Then } \mathbb{P}(v \notin H'') \leq \left(1 - \frac{d}{|H'|}\right)^r \leq \exp\left(-\frac{dr}{|H'|}\right) \leq \exp(-2) < 1/4.$$

$$\Rightarrow \mathbb{P}(uv \in E(H'')) > 1 - 1/4 - 1/4 = 1/2 \quad \text{for } uv \in E(H).$$

$$\Rightarrow \mathbb{E}[e(H'')] > 1/2\, e(H') \geq 1/4\, d|H'|.$$

We also claim that $\chi(H'') \leq r$. Indeed, let $\chi : V(H'') \to [r]$ be defined as $\chi(u) = \min\{i : uv_i \in E\}$.

Suppose that $\chi(u) = \chi(v) = i$ & $uv \in E(H'')$.
Then $uv, uv_i, vv_i \in E(G) \Rightarrow u, v, v_i$ form Δ in G #.
Thus $\chi(H'') \leq r \Rightarrow sp(H'') \geq \Omega\left(\frac{e(H'')}{\chi(H'')}\right)$

$$= \Omega\left(\frac{d|H'|}{|H'|/d}\right) = \Omega(d^2) = \Omega(m^{4/5}).$$

Thus $sp(G) = \Omega(m^{4/5})$ follows from Exercise 32 $\square$

Theorem 34 (Alon) Suppose that G is Δ-free. Then $sp(G) = \Omega(m^{4/5})$, and the exponent $4/5$ is tight.

Proof: Tightness follows from Theorem 24. If G is $m^{2/5}$-degenerate, Lemma 31 implies the result. Otherwise, Lemma 33 implies the result

GW for $disc^+$

Suppose that for each $u \in V(G)$ x_u is a unit vector. Then

$$disc^+(G) \geq \frac{1}{2\pi}\left(\sum_{uv \in E}\left(\frac{\pi}{2} + \arcsin(\langle x_u, x_v\rangle)\right) - \frac{p}{2}\sum_{u \neq v}\left(\frac{\pi}{2} + \arcsin(\langle x_u, x_v\rangle)\right)\right).$$

Note that the constant terms simply cancel out.

Again, we use the approximation

$$disc^+(G) = \Omega\left(\sum_{uv \in E}\langle x_u, x_v\rangle\right) - O\left(p \cdot \sum_{\substack{u \neq v \\ uv \notin E}}\langle x_u, x_v\rangle\right).$$

(Here, we are assuming that $p \leq 1 - \varepsilon$).

Theorem 36 (R., Sudakov, Tomon '2c)

$$disc^+(G) \geq \begin{cases} \Omega(\sqrt{d}\, n) & \text{when } 1 \leq d \leq n^{2/3}. \\ \Omega(n^2/d) & \text{when } n^{2/3} \leq d \leq \frac{1}{2}n. \end{cases}$$

We only prove Thm 36 for regular graphs (Actually, the proof works if $\Delta \leq C \cdot d$, C constant)

Proof: Choose $x_u(v) = \begin{cases} 1 & \text{if } u = v \\ \alpha & \text{if } uv \in E, \ 0 < \alpha \leq 1/\sqrt{d} \text{ to be determined later} \\ 0 & \text{otherwise.} \end{cases}$

Then $\|x_u\|_2^2 = 1 + d\alpha^2 \leq 2$ as $\alpha \leq 1/\sqrt{d}$.

If $uv \in E$, then $\langle x_u, x_v \rangle = 2\alpha + \alpha^2 |N(u) \cap N(v)| \geq 2\alpha$.
Otherwise, $\langle x_u, x_v \rangle = \alpha^2 |N(u) \cap N(v)|$.

Thus $\sum_{uv \notin E} \langle x_u, x_v \rangle \leq \sum_{u \neq v} \alpha^2 |N(u) \cap N(v)| \leq \alpha^2 n d(d-1)$

as each $w \in V$ contributes ~~at one~~ to exactly $d(d-1)$ terms in the sum (for any $(u,v) \in N(w) \times N(w)$ with $u \neq v$)

Thus, $\mathrm{disc}^+(G) = \Omega(\alpha n d) - O(p \cdot n d^2 \alpha^2)$.

$= \Omega(\alpha n d) - O(d^3 \alpha^2)$.

(i) If $d \leq n^{2/3}$, choose $\alpha = {}^c\!/\!\sqrt{d}$ (for sufficiently small constant).

Then $\alpha n d = c \cdot \sqrt{d}\, n$ and $d^3 \alpha^2 = c^2 d^2 = c^2 \sqrt{d} \cdot d^{3/2} \ll c \sqrt{d}\, n$ as $d \leq n^{2/3}$ for c sufficiently small.

(ii) If $n^{2/3} \leq d \leq {}^1\!/\!_2\, n$, set $\alpha = c \cdot {}^n\!/\!d^2$.

Then $\alpha \cdot \sqrt{d} = c \cdot {}^n\!/\!d^{3/2} \leq c \leq 1$ for c sufficiently small.

Also, $\alpha n d = c \cdot n^2/d$ and $d^3 \alpha^2 = c^2 \cdot n^2/d$, so choosing c small enough we get $\mathrm{disc}^+(G) = \Omega(n^2/d)$ $\square$.