In '06, Bollobás and Scott proved the following inverse-relation between disc^- and disc^+.

Theorem 17: (Bollobás, Scott '06). Suppose that $1/n \leq p \leq 1-1/n$. Then $\mathrm{disc}^-(G)\mathrm{disc}^+(G) = \Omega(p(1-p)n^3)$.

Remarks: (i) As $\mathrm{disc}^{\pm}(G) = O(p(1-p)n^2)$, Theorem 17 implies Exercise 16.

(ii) Theorem 17 also implies Theorem 15, and gives a "stability version" of it: If $\mathrm{disc}(G) = \Theta(\sqrt{p(1-p)n^3})$, then $\mathrm{disc}^-(G)$ & $\mathrm{disc}^+(G)$ are both $\Theta(\sqrt{p(1-p)n^3})$.

(iii) Theorem 17 is tight (up to a multiplicative constant) for $G(n,p)$, $T_r(n)$ and $T_r(n)^c$. In addition, there are other non-trivial examples where it is tight, including all strongly regular graphs (see later).

A key step in the proof of Theorem 17 is the following Lemma.

Lemma 18: Suppose $1/n \leq p \leq 1-1/n$. There exists $\beta > 0$ for which the following holds (& which is independent of n & p). Let $V = X \cup Y$ be a random bipartition. Then

$$\mathbb{E}\left[\sum_{x \in X} \big| |N(x) \cap Y| - p|Y| \big| \right] \geq \beta \cdot \sqrt{p(1-p)n^3}.$$

Note that there is nothing surprising on RHS: it is what one expects for binomial random variables. We now prove Theorem 17 assuming Lemma 18.

Proof of theorem 17. Let c be a sufficiently small constant, and suppose that $\mathrm{disc}^+(G) = \frac{c}{\alpha}\sqrt{p(1-p)n^3}$

(wlog $\mathrm{disc}^+(G) \le \mathrm{disc}^-(G)$; if both $\mathrm{disc}^+(G), \mathrm{disc}^-(G) \geq c\sqrt{p(1-p)n^3}$ then the result holds trivially). Let β be the constant given in Lemma 18.

Given a bipartition $X \cup Y$, let $X^+ = \{x \in X : |N(x) \cap Y| \geq p|Y|\}$ & $X^- = X \setminus X^+ = \{x \in X : |N(x) \cap Y| \leq p|Y|\}$.

Given $U, W \subseteq V(G)$ disjoint, define $\mathrm{disc}(U,W) = e(U,W) - p|U||W|$ and note that $\mathrm{disc}(U,W) = \sum_{u \in U} \left(|N(u) \cap W| - p|W| \right)$

Note that $\mathrm{disc}(X^+, Y) - \mathrm{disc}(X^-, Y)$.

$= \sum_{x \in X} \left| |N(x) \cap Y| - p|Y| \right|$ (as every individual term in X^+ is ≥ 0 & in X^- is ≤ 0). without absolute values!

and $\mathrm{disc}(X^+, Y) + \mathrm{disc}(X^-, Y) = \mathrm{disc}(X^+ \cup X^-, Y) = \mathrm{disc}(X, Y)$.

Thus, whenever $X \cup Y$ is a random bipartition, we have

$\mathbb{E}[\mathrm{disc}(X,Y)] = 0$ and

$\mathbb{E}\left[\sum_{x \in X} \left| |N(x) \cap Y| - p|Y| \right| \right] \geq \beta\sqrt{p(1-p)n^3}$ (Lemma 18)

So $\mathbb{E}[\mathrm{disc}(X^+, Y)] \geq \beta/2 \sqrt{p(1-p)n^3}$ $(*)$

Let W be a random subset of X^+ chosen so that each $w \in X^+$ belongs to W with probability q (and these choices are made independently).

Note that $\mathbb{E}[\mathrm{disc}(W)] = q^2 \mathrm{disc}(X^+)$
and $\mathbb{E}[\mathrm{disc}(W, Y)] = q\, \mathrm{disc}(X^+, Y)$.

Let $X \cup Y$ be chosen so that

$$\mathrm{disc}(X^+, Y) + \alpha\, \mathrm{disc}(Y) \geq \beta/2 \sqrt{p(1-p)n^3},$$

and note that such choices exist as by $(*)$ & $\mathbb{E}[\mathrm{disc}(Y)] = 0$ the expectation of LHS = RHS.

For these choices of X & Y, and for $q = 1/\alpha$, we conclude that

$$\mathbb{E}[\mathrm{disc}(W \cup Y)] = \frac{1}{\alpha^2}\mathrm{disc}(X^+) + \frac{1}{\alpha}\mathrm{disc}(X^+, Y) + \mathrm{disc}(Y)$$
$$\geq 1/\alpha^2\, \mathrm{disc}(X^+) + \frac{\beta}{2\alpha}\sqrt{p(1-p)n^3}.$$

Since $\mathrm{disc}(W \cup Y) \leq \mathrm{disc}^+(G) = \frac{c}{\alpha}\sqrt{p(1-p)n^3}$ for all choices of W & Y, by choosing $c = \beta/4$ we conclude that $\mathrm{disc}(X^+) \leq -\frac{\beta}{4}\alpha\sqrt{p(1-p)n^3}$.

Thus $\mathrm{disc}^-(G) \geq \beta/4\, \alpha \sqrt{p(1-p)n^3}$, and hence
$\mathrm{disc}^-(G)\,\mathrm{disc}^+(G) \geq \beta^2/16\; p(1-p)n^3$ $\square$.

Proof of Lemma 18: Wlog assume that $p \leq 1/2$.

Now, $\mathbb{E}\left[\sum_{x\in X} \big||N(x)\cap Y| - p|Y|\big|\right] = \mathbb{E}\left[\sum_{x\in V} \mathbb{1}\{x\in X\} \big||N(x)\cap Y| - p|Y\setminus x|\big|\right]$.

$= \frac{1}{2}\sum_{x\in V} \mathbb{E}\left[\big||N(x)\cap Y| - p|Y\setminus x|\big|\right]$.

Fix $x\in V$, and write $r(x) = d(x) - p(n-1)$.
Given $v\neq x$, write $e_v = \mathbb{1}\{xv\in E\}$, $p_v = \mathbb{1}\{v\in Y\}$
& $\varepsilon_v = 2p_v - 1 \in \{\pm 1\}$. Thus ε_v's are iid ± 1 rvs.

Now, $\big||N(x)\cap Y| - p|Y\setminus x|\big| = \left|\sum_{v\neq x} p_v(e_v - p)\right|$.

$= \left|\underbrace{\frac{1}{2}\sum_{v\neq x}(e_v - p)}_{=r(x)} + \frac{1}{2}\sum_{v\neq x}\varepsilon_v(e_v-p)\right|$.

Thus $\mathbb{E}\left[\big||N(x)\cap Y| - p|Y\setminus x|\big|\right] \geq \frac{1}{2}\max\left\{ \cancel{\mathbb{E}}\,|r(x)|, \mathbb{E}\left[\left|\sum_{v\neq x}\varepsilon_v(e_v - p)\right|\right]\right\}$

as distribution of $\sum_{v\neq x}\varepsilon_v(e_v-p)$ is symmetric about 0.

Suppose that $N(x) = \{v_1, \dots, v_d\}$. Thus & $v_n = x$, $d = d(x)$

$\mathbb{E}\left[\left|\sum_{v\neq x}\varepsilon_v(e_v-p)\right|\right] = \mathbb{E}\left[\left|\sum_{i=1}^{d}\varepsilon_i(1-p) + \sum_{i=d+1}^{n-1} -p\cdot\varepsilon_v\right|\right]$.

$\geq \mathbb{E}\left[\left|\sum_{i=1}^{d}\varepsilon_i(1-p)\right|\right]$ as distribution of $\sum_{i=d+1}^{n-1} -p\varepsilon_v$ is symmetric about 0,

$\approx (1-p)\cdot\sqrt{d/2} = (1-p)\sqrt{d(x)/2}$.

Thus $\mathbb{E}\left[\big||N(x)\cap Y| - p|Y\setminus x|\big|\right] \geq c\cdot\left(|r(x)| + (1-p)\cdot\sqrt{d(x)/2}\right)$
and since $p\geq 1/n$, $p\leq 1/2$, this is $\geq c'\sqrt{p(n-1)}$. for some constant c'.
The result follows by adding these n terms up □.

We now discuss some similarities & differences of $\mathrm{disc}^{\pm}(G)$, $sp(G)$, λ_2, λ_n & $bw(G)$.
Here, $bw(G)$, the bisection width of G, is defined as

$$bw(G) = \min_{\substack{A \cup B = V \\ A \cap B = \phi \\ ||A| - |B|| \le 1}} e(A,B) \qquad (\text{"Min-cut" among equipartitions}),$$

$$dfc(G) = m/2 - e(A,B) \qquad (\text{deficit of } G).$$

and $\mathrm{disc}(U,W) := e(U,W) - p|U||W|$.

Exercise 19: (i) Prove that $\mathrm{disc}(U) + \mathrm{disc}(U,W) + \mathrm{disc}(W) = \mathrm{disc}(U \cup W)$.

(ii) Suppose that $1/n \le p \le 1 - 1/n$. Prove that $\mathrm{disc}^-(G) = \Omega(sp(G))$ & $\mathrm{disc}^+(G) = \Omega(dfc(G))$.

(iii) Prove that the converse statements need not hold.

(iv) Suppose that G is regular, & $1 \le d \le n-2$. Prove that $sp(G) = \Theta(\mathrm{disc}^-(G))$ & $dfc(G) = \Theta(\mathrm{disc}^+(G))$.

(v) Let G be d-regular. Prove that $\forall U \subseteq V(G)$,

$$\frac{\lambda_n}{2}|U| - p|U| \le \mathrm{disc}(U) \le \lambda_2/2\,|U| + p|U|.$$

Corollary 20: $(\lambda_2 + 1)|\lambda_n| = \Omega(d)$ whenever G is d-regular & $d \le (1-\varepsilon)n$

Proof: Follows from Exercise 19(v) + Bollobás-Scott $\square$.

Exercise 21: Prove that Bollobás-Scott is tight (up to multiplicative constants) for strongly regular graphs.

Recall that $\mathrm{disc}^+(G) = \Omega(n)$, $\mathrm{disc}^-(G) = \Omega(n)$, and these are tight for Turan graphs / their complements. While sp is unaffected by adding isolated vertices, $\mathrm{disc}^\pm$ are not. Thus, in some sense $\mathrm{disc}^\pm$ are more sensitive. As Turán graphs only exist for certain densities $1 - 1/r$, it remains unclear whether $\Omega(n)$ is still the correct lower bound say when $p = 2/5$.

Conjecture 22 (Verstraëte, stated informally). If G is "far away from Turan graph" and has density p, then $\mathrm{disc}^+(G) = \Omega(\mathrm{disc}^+(G_{n,p}))$.

Theorem 23 (Alon, '91). If G is d-regular and $d = O(n^{1/9})$, then $\mathrm{disc}^+(G) = \Omega(\sqrt{d}\, n)$.

Theorem 24 (R., Sudakov, Tomon 26).
If G has average degree d, then

$$\mathrm{disc}^+(G) = \begin{cases} \Omega(n\sqrt{d}) & \text{if } 1 \leq d \leq n^{2/3}; \text{ this is tight.} \\ \Omega(n^2/d) & \text{if } n^{2/3} \leq d \leq n^{4/5}; \text{ tight for } d \leq n^{3/4}. \\ \Omega(d^{1/4} n / \log n) & \text{if } n^{4/5} \leq d \leq (1/2 - \varepsilon) n. \end{cases}$$

We discuss the proof later in the course.

Theorem 25: (Zhang 25+; see also Jin, Milojević, Tomon, Zhang 25+). (Informally stated). There exists $\delta > 0$ so that if G is far away from $T_r(n)$ then $\mathrm{disc}^+(G) = \Omega(n^{1+\delta})$.

Goemans - Williamson algorithm:

Note that

$$mc(G) = \max_{x_1,\dots,x_n \in \{-1,1\}} \sum_{ij \in E} \tfrac{1}{2}(1 - x_i x_j) = \max_{x_1,\dots,x_n \in [-1,1]} \sum_{ij \in E} \tfrac{1}{2}(1 - x_i x_j)$$

where the first equality follows by taking $A = \{i : x_i = -1\}$, $B = \{i : x_i = 1\}$ and the second follows from the fact that the function we are optimizing is linear in each x_i (and thus maximized at the endpoint of the interval).

Define $mc^*(G)$ to be the semidefinite relaxation of $mc(G)$, ie.

$$mc^*(G) = \sup_{\substack{v_1,\dots,v_n \in \mathbb{R}^N \\ \|v_i\|_2 \leq 1}} \sum_{ij \in E} \tfrac{1}{2}\left(1 - \langle v_i, v_j \rangle\right).$$

It is well-known that $mc(G)$ is inefficient to compute, but $mc^*(G)$ can be determined in polynomial time.

Theorem 26 Let α be a constant chosen so that $\arccos(z) \geq \alpha(1-z)$ for all $z \in [-\pi/2, \pi/2]$. Then

$$mc^*(G) \geq mc(G) \geq \frac{\alpha}{\pi} mc^*(G)$$

(RHS is $\approx 0.868\, mc^*(G)$).

Remarks: (i) Thus $mc(G)$ can be approximated to (a decent) constant factor in polynomial time.

(ii) Under certain assumptions (including assuming some central open problems, such as unique games conjecture), the constant α/π cannot be improved for an approximate algorithm that can be computed in polynomial time.

(iii) On its own, not useful to lower bound $sp(G)$ algorithmically. But the methods used in the proof apply to $sp(G)$ as well.

Theorem 27 Let G be a graph, and suppose for each $u \in \mathbb{R}^N$ we assign a non-zero vector $x_u \in \mathbb{R}^N$. Then

$$sp(G) \geq -\frac{1}{\pi} \sum_{uv \in E} \arcsin\left(\frac{\langle x_u, x_v\rangle}{\|x_u\|_2 \|x_v\|_2}\right) \qquad (*)$$

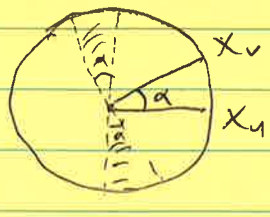
If x_u is always an unit vector, then the term above simplifies to $\arcsin(\langle x_u, x_v\rangle)$. Sometimes, it turns out to be more convenient to work with vectors x_u that do not necessarily have equal length, and hide the rounding error to multiplicative constants.

Proof: Pick $w \in S^{N-1}$ uniformly at random, and set $A = \{u : \langle x_u, w\rangle \geq 0\}$, $B = V \setminus A$. We claim that $\mathbb{E}[sp(A,B)] = \text{RHS of } (*)$, and that each $uv \in E$ contributes $-\frac{1}{\pi}\arcsin\left(\frac{\langle x_u, x_v\rangle}{\|x_u\| \|x\|_v}\right)$ to the sum.

Let $\alpha = \arccos\left(\frac{\langle x_u, x_v\rangle}{\|x_u\|_2 \|x_v\|_2}\right)$, and thus $\arcsin\left(\frac{\langle x_u, x_v\rangle}{\|x_u\| \|x_v\|}\right) = \frac{\pi}{2} - \alpha$.

Then $\mathbb{P}(uv \text{ is in cut}) = \frac{2\alpha}{2\pi} = \frac{\alpha}{\pi} = \frac{1}{2} - \frac{1}{\pi}\arcsin\left(\frac{\langle x_u, x_v\rangle}{\|x_u\| \|x_v\|}\right)$

as writing $w = w_0 + w_1$, $w_0 \in \operatorname{Span}\{x_u, x_v\}$, $w_1 \in \operatorname{Span}\{x_u, x_v\}^{\perp}$, then (Conditionally on $w_0 \neq 0$), $w_0/\|w_0\|_2$ is uniformly distributed on S^1,



Shaded region is the "suitable region" for w_0.

Thus, the result follows as

$$\mathbb{E}[sp(A,B)] = \sum_{uv \in E}\left(\frac{1}{2} - \frac{1}{\pi}\arcsin\left(\frac{\langle x_u, x_v\rangle}{\|x_u\|_2 \|x_v\|_2}\right)\right) - \frac{m}{2} = -\frac{1}{\pi}\sum_{uv \in E}\arcsin\left(\frac{\langle x_u, x_v\rangle}{\|x_u\|_2 \|x_v\|_2}\right)$$