Removal Lemmas: Summer School 2025

1 The regularity and removal lemmas

The graph removal lemma is the following statement:

Theorem 1.1 (Graph removal lemma, Ruzsa-Szemerédi '78)**.** *Let H be a fixed graph. For every $\varepsilon > 0$ there is $\delta = \delta_H(\varepsilon) > 0$ such that if an n-vertex graph G has at most $\delta n^{v(H)}$ copies of H, then G can be made H-free by deleting at most εn^2 edges.*

Remarks:

- We say that G is *ε-far* from being H-free if one has to delete at least εn^2 edges to turn G into an H-free graph. The contrapositive is that if G is ε-far from H-free then G has at least $\delta n^{v(H)}$ copies of H.
- Being ε-far from H-free is equivalent to having a collection of $\Theta(\varepsilon)n^2$ edge-disjoint copies of H. Indeed, if G has such a collection of size εn^2, then G is ε-far (because we have to delete at least one edge from each H-copy in order to destroy all H-copies in G). In the other direction, take a maximal collection of edge-disjoint copies of H in G. Deleting all edges of these copies makes the graph H-free (because of the maximality of the collection). Thus, if the maximal such collection has size less than $\frac{\varepsilon}{e(H)}n^2$, then G is not ε-far.

The removal lemma is proved using Szemerédi's regularity lemma, which we now recall. Consider a bipartite graph with parts X, Y. The *density* is $d(X,Y) := \frac{e(X,Y)}{|X||Y|}$.

Definition 1.2 (Regular pair)**.** *A bipartite graph (X,Y) is* ε-regular *if for every $X' \subseteq X, Y' \subseteq Y$ with $|X'| \geq \varepsilon|X|, |Y'| \geq \varepsilon|Y|$, it holds that $|d(X',Y') - d(X,Y)| \leq \varepsilon$.*

Regular pairs are "random-like". Indeed, the definition captures a key property of random graphs: uniform edge distribution. Another key random-like property of regular pairs is given by the counting lemma:

Lemma 1.3 (Counting lemma)**.** *For every $\gamma > 0$ there is $\varepsilon > 0$ such that if $V_1, \dots, V_r$ are disjoint vertex sets such that all pairs (V_i, V_j) are ε-regular, then the number of r-cliques $v_1, \dots, v_r$ (with $v_i \in V_i$) is*

$$\prod_{i=1}^{r} |V_i| \cdot \left(\prod_{1 \leq i < j \leq r} d(V_i, V_j) \pm \gamma \right). \tag{1}$$

Note that (1) (with the error γ omitted) is precisely the expected number of r-cliques if the edges between V_i and V_j were chosen randomly with probability $d(V_i, V_j)$, for every $1 \leq i < j \leq r$. In many applications, it suffices to have a lower bound for the number of r-cliques. To illustrate how the proof of the counting lemma works, let us prove such a statement in the case $r = 3$ (the proof for general r is similar, via induction). We will assume that all densities $d(V_i, V_j)$ are large enough in terms of ε.[1]

Lemma 1.4. *For every $d > 0$ there is $\varepsilon = d/2$ so that if V_1, V_2, V_3 are such that $d(V_i, V_j) \geq d$ and (V_i, V_j) is ε-regular for every $1 \leq i < j \leq 3$, then there are at least $(1-2\varepsilon)(d-\varepsilon)^3|V_1||V_2||V_3| \geq (d^3 - 5\varepsilon)|V_1||V_2||V_3|$ triangles.*

Proof. First we need the following simple property of regular pairs. The proof is left to the reader.

Claim 1.5. *Let (X, Y) be an ε-regular pair with density $d = d(X, Y)$. Then at most $\varepsilon|X|$ of the vertices $x \in X$ satisfy $\frac{d_Y(x)}{|Y|} < d - \varepsilon$, and at most $\varepsilon|X|$ of the vertices $x \in X$ satisfy $\frac{d_Y(x)}{|Y|} > d + \varepsilon$.*

Now we prove Lemma 1.4. For $i = 1, 2$, let B_i be the set of vertices $v \in V_3$ with $d_{V_i}(v) < (d-\varepsilon)|V_i|$. By Claim 1.5, we have $|B_i| \leq \varepsilon|V_3|$. So $|B_1 \cup B_2| \leq 2\varepsilon|V_3|$. For each $v \in V_3 \setminus (B_1 \cup B_2)$, consider $U_1 := N_{V_1}(v)$ and $U_2 := N_{V_2}(v)$. As $v \notin B_1 \cup B_2$, we have $|U_1| \geq (d-\varepsilon)|V_1| \geq \varepsilon|V_1|$ and similarly $|U_2| \geq \varepsilon|V_2|$. By the regularity of (V_1, V_2), we have $d(U_1, U_2) \geq d - \varepsilon$, and therefore $e(U_1, U_2) \geq (d-\varepsilon)|U_1||U_2| \geq (d-\varepsilon)^3|V_1||V_2|$. Each edge in $E(U_1, U_2)$ creates a triangle with v. Doing this for all (at least $(1-2\varepsilon)|V_3|$) choices of $v \in V_3 \setminus (B_1 \cup B_2)$, we get at least $(1-2\varepsilon)(d-\varepsilon)^3|V_1||V_2||V_3|$ triangles, as required. ∎

Another version of the counting lemma we will use is as follows.

Definition 1.6 (bi-induced copy)**.** *A* bi-induced copy *of a bipartite graph $H = (A, B)$ in a graph G is an injection $\varphi : V(H) \to V(G)$ such that for every $a \in A, b \in B$, $ab \in E(H)$ if and only if $\varphi(a)\varphi(b) \in E(G)$. If G is itself bipartite with parts X, Y, then we also require that $\varphi(A) \subseteq X$ and $\varphi(B) \subseteq Y$.*

Note that in the above definition we do not make requirements on the edges inside $\varphi(A)$ and $\varphi(B)$.

Lemma 1.7. *For every integer k and $d > 0$, there is $\varepsilon > 0$ such that the following holds. Consider a bipartite graph (X, Y) and suppose that $d \leq d(X, Y) \leq 1 - d$ and (X, Y) is ε-regular. Then (X, Y) contains a bi-induced copy of every bipartite graph (A, B) with $|A|, |B| \leq k$.*

One can deduce the above lemma from Lemma 1.3 as follows: Suppose that $A = \{a_1, \dots, a_k\}$, $B = \{b_1, \dots, b_k\}$. Split X into equal parts $X_1, \dots, X_k$ and Y into equal parts $Y_1, \dots, Y_k$. Define an auxiliary graph as follows: If $a_ib_j \in E$ then take the edges of G between X_i, Y_j, and if $a_ib_j \notin E$ then take the non-edges of G between X_i, Y_j. Now apply Lemma 1.3 to this auxiliary graph.

The Szemerédi regularity lemma states that any graph has a vertex partition into a bounded number of parts, such that most pairs of parts are regular.

Theorem 1.8 (Szemerédi's regularity lemma 1978)**.** *For every $\varepsilon > 0$ and $t_0 \geq 1$, there is $T = T(\varepsilon, t_0)$ such that the following holds. Every graph G on $n \geq T$ vertices has an equipartition*[2] *$V(G) = V_1 \cup \dots \cup V_t$ with $t_0 \leq t \leq T$ such that all but εt^2 of the pairs (V_i, V_j), $1 \leq i < j \leq t$, are ε-regular.*

An equipartition as in Theorem 1.8 is called *ε-regular*. Let us give a very rough sketch of the proof of the regularity lemma.

[1] Otherwise, i.e. if some $d(V_i, V_j)$ is smaller than γ, then it is easy to see that the statement of Lemma 1.3 holds trivially (because the number of r-cliques is at most $|V_1| \dots |V_r| d(V_i, V_j)$).

[2] An equipartition is a partition in which any two parts V_i, V_j satisfy $||V_i| - |V_j|| \leq 1$.

Proof sketch of Theorem 1.8. For a partition $\mathcal{P} = \{V_1, \ldots, V_t\}$ of $V(G)$, we define the *mean square density* as

$$q(\mathcal{P}) = \sum_{1 \le i < j \le t} \frac{|V_i||V_j|}{n^2} d^2(V_i, V_j).$$

One shows that:

1. If $\mathcal{Q}$ is a refinement of $\mathcal{P}$ then $q(\mathcal{Q}) \ge q(\mathcal{P})$.

2. If $\mathcal{P}$ is not ε-regular, then there is a refinement $\mathcal{Q}$ of $\mathcal{P}$ with $q(\mathcal{Q}) \ge q(\mathcal{P}) + \varepsilon^5$ and $|\mathcal{Q}| \le 2^t \cdot t$, where $t = |\mathcal{P}|$.

A proof sketch for Item 2 is as follows: For each pair $1 \le i < j \le t$ such that (V_i, V_j) is not ε-regular, take $V_{ij} \subseteq V_i, V_{ji} \subseteq V_j$ such that $|V_{ij}| \ge \varepsilon |V_i|$, $|V_{ji}| \ge \varepsilon |V_j|$, and $|d(V_{ij}, V_{ji}) - d(V_i, V_j)| > \varepsilon$. Now, for each $1 \le i \le t$, take the common refinement (Venn diagram) of all sets $(V_{ij} : j)$. The resulting partition is Q. It is easy to see that $|\mathcal{Q}| \le 2^t \cdot t$, and one can show that $q(\mathcal{Q}) \ge q(\mathcal{P}) + \varepsilon^5$.

Using Items 1-2, one proves Theorem 1.8 as follows. Start with an arbitrary equipartition $\mathcal{P}_0$ into t_0 parts. If $\mathcal{P}_i$ is not ε-regular, use Item 2 to get a refinement $\mathcal{P}_{i+1}$ with $q(\mathcal{P}_{i+1}) \ge q(\mathcal{P}_i) + \varepsilon^5$. As $q(\mathcal{P}) \le 1$ for any $\mathcal{P}$, the process has to stop in at most $\frac{1}{\varepsilon^5}$ steps.

At each iteration, there is also an additional step of turning the partition $\mathcal{Q}$ given by Item 2 into an equipartition, by chopping up the parts of $\mathcal{Q}$ into equal-sized sets. One can show that if the set-size is small enough, this does not decrease $q(\mathcal{Q})$ by much. ■

What is the bound on the partition-size T that we get in Theorem 1.8? The proof is via a procedure that runs for poly$(1/\varepsilon)$ steps, and at each step we replace a partition of size t with a partition of size roughly 2^t. Hence, the number of parts is at most tower(poly$(1/\varepsilon), t_0)$, where

$$\text{tower}(k, x) = 2^{2^{2^{\cdot^{\cdot^{\cdot^{2^x}}}}}} \Big\} k \text{ times}$$

I.e., the bound is of tower type. Gowers proved that this is inevitable.

Theorem 1.9 (Gowers 1997)**.** *There are graphs which require* tower(ε^{-c}) *parts in any ε-regular partition, where $c > 0$ is a constant.*

Let us now prove the removal lemma (Theorem 1.1). For simplicity, we consider the case $H = K_3$.

Proof of the triangle removal lemma. Let G be a graph which is γ-far from K_3-free. Apply the regularity lemma with parameters $\varepsilon = \gamma/10$ and $t_0 = 10/\gamma$ to obtain an ε-regular partition $V_1, \ldots, V_t$ with $t_0 \le t \le T$. We now clean the graph. I.e., we delete the following edges:

1. All edges inside V_i for every $1 \le i \le t$.

2. All edges between pairs (V_i, V_j) with $d(V_i, V_j) \le 2\varepsilon$.

3. All edges between pairs (V_i, V_j) which are not ε-regular.

The number of edges of type 1 is at most $t \cdot \binom{n/t}{2} \le \frac{n^2}{t} \le \frac{\gamma}{10} n^2$. The number of edges of type 2 is at most $2\varepsilon \cdot \sum_{1 \le i < j \le t} |V_i||V_j| \le 2\varepsilon \binom{n}{2} \le \frac{\gamma}{5} n^2$. The number of edges of type 3 is at most $\varepsilon t^2 \cdot \left(\frac{n}{t}\right)^2 = \varepsilon n^2 = \frac{\gamma}{10} n^2$. So the total number of deleted edges is less than γn^2. As G is γ-far from K_3-free, the remaining graph (after the deletion of these edges) still has a triangle. This triangle cannot contain two vertices from the same part V_i (because of Item 1). So suppose that this triangle has one vertex in each of the sets V_i, V_j, V_k. Then by Items 2-3, all pairs $(V_i, V_j), (V_i, V_k), (V_j, V_k)$ are ε-regular and have density at least 2ε. By Lemma 1.4, there are at least $\text{poly}(\varepsilon)|V_i||V_j||V_k|$ triangles. Now, $|V_i||V_j||V_k| = (n/t)^3 \geq n^3/T^3$, so we can set $\delta := \frac{\text{poly}(\varepsilon)}{T^3}$. ■

Note that because of the tower-type parameter dependence in the regularity lemma, the above proof gives a tower-type dependence for the removal lemma as well. Namely, it shows that in Theorem 1.1, we can take $1/\delta = \text{tower}(\text{poly}(1/\varepsilon))$. This was improved to $\text{tower}(O(\log \frac{1}{\varepsilon}))$ by Fox. It is a major open problem to improve this further.