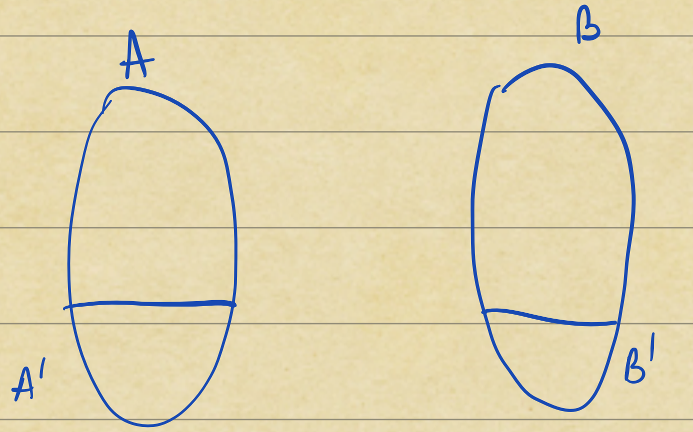
Key Matching Lemma: Let $G = (A, B, E)$ be ε-super-regular. Then there exists a distribution λ on perfect matchings of G such that λ is $O_{\delta,\varepsilon}(1/n)$-spread.

Proof: Consider a random subgraph K of G where we include a random
(independent)
C edges around each vertex.

We want to find a perfect matching in K.



Let $d =$ density of G.

Consider $A' \subseteq A$, $|A'| \leq \frac{|A|}{2}$, $|B'| = |A'| - 1$. Let $|A'| = x$.

Case 1: $x \geq \eta n$:

By regularity, $\#\{v : |N_G(v) \cap B'| > \left(\frac{x}{n} + \frac{\varepsilon}{d}\right) dn\}$
$\leq \varepsilon n$.

$\Rightarrow$ At least $x - \varepsilon n$ $v \in A'$ has $|N_G(v) \cap B'| \leq \left(\frac{x}{n} + \frac{\varepsilon}{d}\right) dn$.

$\mathbb{P}(N_K(v) \leq B') \leq \left(\frac{x}{n} + \frac{\varepsilon}{d}\right)^C$, and so

$$\mathbb{P}(N_K(A) \subseteq B') \leq \left(\frac{x}{n} + \frac{\varepsilon}{d}\right)^{C(x - \varepsilon n)}.$$

By the union bound over A', B',

$$\mathbb{P}(\exists |A'| = x,\ |N_K(A')| < |A'|) \leq \binom{n}{x}\binom{n}{x-1}\left(\frac{x}{n}+\frac{\varepsilon}{d}\right)^{C(x-\varepsilon n)}$$

$$\leq \left(\frac{e^2n^2}{x^2}\right)^x \left(\frac{x}{n}+\frac{\varepsilon}{d}\right)^{Cx/2}$$

$$= \left(\frac{e^2n^2}{x^2}\cdot\left(\frac{x}{n}+\frac{\varepsilon}{d}\right)^{C/2}\right)^x.$$

$$\leq \left(\frac{e^2n^2}{x^2}\cdot\left(\frac{x}{n}+\frac{\varepsilon}{d}\right)^2\cdot\left(\frac{2}{3}\right)^{C/2-2}\right)^x$$

$$< 2^{-x}.$$

Case 2: $x < \eta n$.

For every v, $\mathbb{P}(N_K(v) \subseteq B') \leq \left(x/dn\right)^C$.

By union bound,

$$\mathbb{P}(\exists A' : |A'| \leq \eta n,\ |N_K(A')| < |A'|)$$

$$\leq \sum_{x \leq \eta n} \binom{n}{x}\binom{n}{x-1}\left(x/dn\right)^{Cx}$$

$$\leq \sum_{x \leq \eta n}\left(\frac{e x^{C-2}}{d^C n^{C-2}}\right)^x < \sum_{x \leq \eta n}\left(\frac{e\eta^{C-2}}{d^C}\right)^x = o(1).$$

Let λ be the distribution of an arbitrary perfect matching in K (if exist).

$$\lambda(\langle\{a_1,b_1\},\dots,\{a_t,b_t\}\rangle) \leq \frac{1}{1-\mathbb{P}(K \text{ has a p.m.})}\cdot\mathbb{P}(\{a_i,b_i\}\in E(K))$$

$$\leq (1+o(1))\quad\left(2C/n\right)^t. \qquad \square$$

Lem: Let $G[V_1, \dots, V_r]$ be s.t. $|V_i| = n \; \forall i$, $G[V_i, V_j]$ (ε, δ)-super-regular.

Let $\mathcal{H}_G$ be the hypergraph whose edges are the copies of K_r in G.

Then $\exists \; O(n^{1-r})$-spread distribution on perfect matchings of H.

($\Longrightarrow$ Counting Version)

* This can be deduced directly from the random blow-up lemma yielding vertex-spread K_r-factors. We give below an alternative simpler proof by directly iterating the key matching lemma.

Thm (Riordan, Heckel): $\exists \varepsilon$: for $p \le n^{-2/r + \varepsilon}$, $\exists$ coupling of G_p and $(\mathcal{H}_G)_\pi$, $\pi = p^{\binom{r}{2}}$ where for every hyperedge of $(\mathcal{H}_G)_\pi$, the corresponding clique appears in G_p.

Lem: Let $G[V_1, V_2, V_3]$ be (ε, δ)-super-regular with density d.

Assume $\forall$ edge $\{v_2, v_3\}$, $\exists \; (d^2 - \varepsilon^{1/2}) n$ $v_1 \in V_1$ making a Δ with v_2, v_3.

Let μ_1 be a Cn-spread p.m. between V_2, V_3.

For matching M_1 between V_2, V_3, let Γ_{M_1} be a graph between V_1 and M_1 where $v_1 \sim e_1$ if v_1 is adjacent to e_1 endpoints.

Then Γ_{M_1} contains a $O(C) / \log(1/\varepsilon)^{1/4}$-super-regular subgraph w.p. $1 - \exp(-cn)$.

Pf: Claim: Γ_{M_1} is $4\sqrt{\eta}$-regular, $\eta = (\log 1/\varepsilon)^{-1/2}$.

For each $S_1 \subseteq V_1$ of size ρn, $\rho \ge 4\sqrt{\eta}$, say $e \in G[V_2, V_3]$ is bad if its common neighbors in S is not $(d^2 \pm \eta)\rho n$. Say v_2 / v_3 is bad if its neighborhood in S is not $(d \pm \eta)\rho n$.

$\#$bad $v \le \varepsilon n$; for each good v $\#$bad edges adjacent to $v \le \varepsilon n$.

Say M_1 is bad if it contains $\ge \eta n$ bad edges.

$$\mathbb{P}(M_1 \text{ bad}) \le \left(\frac{C}{n}\right)^n \binom{n}{\eta n} (\varepsilon n)^{\eta n} \, n^{n-\eta n} \le \exp(-(\log 1/\varepsilon)^{1/2} n/4).$$

$\Rightarrow$ Γ_{M_1} $4\eta^{1/2}$-regular whp by union bound.

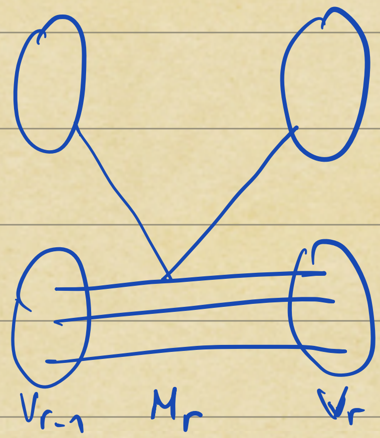
Min-degree for M_1-vertices from assumption.

For each v_1: each v_2/v_3 adjacent to $(d^2 - \varepsilon^{1/2})n$ edges in $N(v_1)$.

$$\mathbb{P}\left(\deg_{\Gamma_{M_1}}(v_1) < \left(\frac{d^2}{2e^2}\right)^C n\right) \le \exp\left(-n \left(\frac{d^2}{2e^2}\right)^C\right).$$

$\square$

Lem follows by induction on r:



Lem: Let H k-uniform be s.t. $\delta_\ell(H) \geqslant (\delta_{\ell k} + \varepsilon)\binom{n}{k-\ell}$. Then $\exists$ $O(n^{1-r})$-spread distribution on perfect matchings of H.

Iterative absorption: $V(H) = V_0 \supseteq V_1 \supseteq \dots \supseteq V_N = X$,

- $|V_{i+1}| = (1 \pm \varepsilon^2/n^2)\,\varepsilon^2\, |V_i|$,
- $|X| \in [n^{1/(k+2)}, n^{1/(k+1)}]$,
- $\forall S \in \binom{V(H)}{\ell}$, $\deg(S, V_i) \geqslant (\delta_{k,\ell} + \varepsilon/2)\binom{|V_i|}{k-\ell}$,
- $\forall$ $v_1, \dots, v_m$, $x_1, \dots, x_m$:
$$\mathbb{P}(v_i \in V_{x_i} \ \forall i) \leq \prod_i \frac{2|V_{x_i}|}{n}.$$

(Random $V_1 \supseteq \dots \supseteq V_N$)

$\forall \varepsilon, \gamma$, $\exists \gamma'$

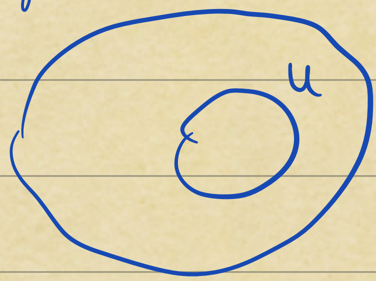
Lem: H with $\delta_\ell(H) \geqslant (\delta_{\ell k} + \varepsilon)\binom{n}{k-\ell}$ then $\exists$ subgraph H' with $\gamma'\binom{n}{k}$ edges and max-deg $(1+\gamma)\gamma'\binom{n}{k-1}$.

Use Ferber-Kwan: If most ℓ-tuples are in $\geqslant (\delta_{\ell,k} + \varepsilon)\binom{n}{k-\ell}$ edges, In random partition, whp min ℓ-deg condition preserved in most parts $\Rightarrow$ H has an almost perfect matching.

Remove matchings iteratively for some $\gamma'\binom{n}{k-1}$ times preserves most ℓ-degrees.

This gives H'.

Lem:



$\forall |S| = \ell$, $\deg(S, U) \geqslant (\delta_{\ell k} + \varepsilon/3)\binom{|U|}{k-\ell}$,

$\exists$ $O(n^{1-k})$-spread matching M covering $V \setminus U$ and

$\leq \epsilon^2 |U|$ vtx of U.

Pf: After finding nearly regular subgraph of $H[V \setminus U]$,

Rodl nibble $\Rightarrow$ spread almost perfect matching.

For remaining vtx, cover by random k-edge with U.

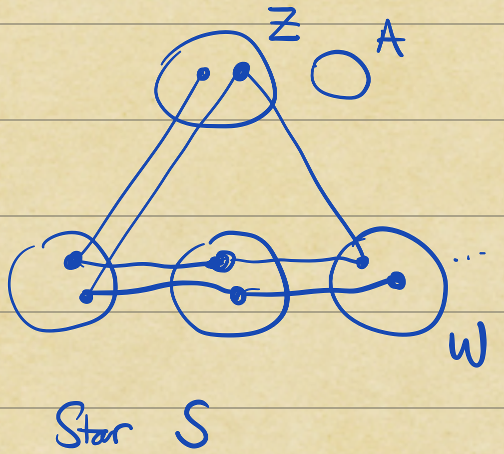
To complete the perfect matching, use min-degree to output an arbitrary matching.

Perturbed Random Graph:

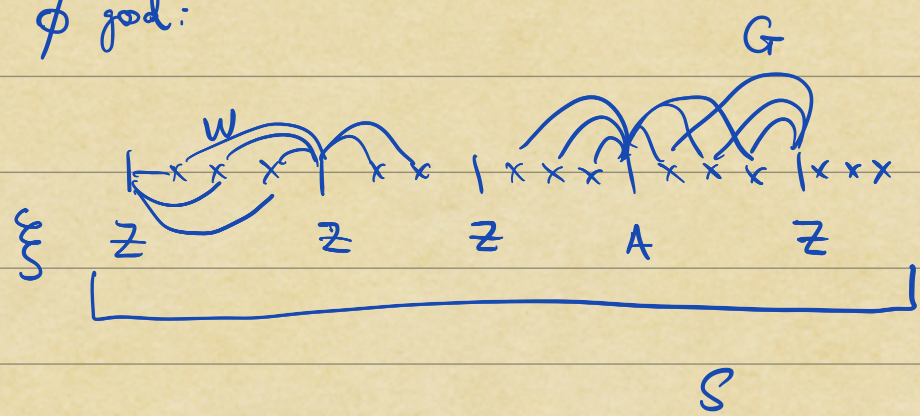
Thm (Nenadov-P. 24): For $k \geq 3$, $\delta(G) \geq \left(\frac{1}{k+1} + \varepsilon\right) n$, then $G \cup G(n,p)$ contains a k^{th} power of the cycle for $p \geq Cn^{-1/(k-1)}$.

Sketch: Apply regularity lemma to G, find in reduced graph vtx disjoint stars of at most k leaves.

Repartitioning $\longrightarrow$ Assume, apart from exceptional part V_0, the rest are in equal-size k-stars.

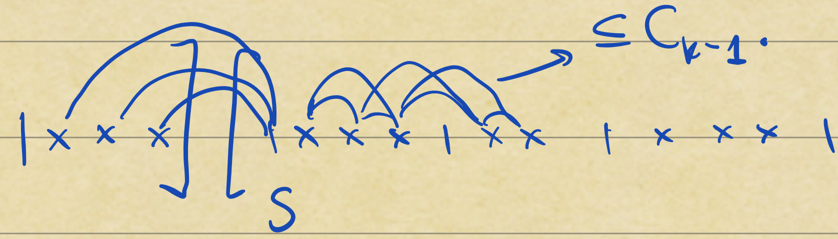


ϕ good:



Random Blow-up Lemma $\Rightarrow \exists\ O(1/n)$-spread distribution on good ϕ.

Now we aim to find $\overset{H_\phi}{\overset{\shortparallel}{E(C_k \setminus \phi)}}$ from the random graph $G(n,p)$.



Lem: λ on H_ϕ is smooth-spread: $\forall$ good ϕ', $H' \subseteq H_{\phi'}$ of h edges, $t \geq hn^{-\delta}$: $\lambda(\{H_\phi : |E(H') \cap E(H_\phi)| = t\}) \leq q^t$, $q = O(n^{-1/(k-1)})$.