Extremal Combinatorics: If G has minimum degree $\delta \geq f(n)$, then G contains H as a subgraph.

Robust Threshold: For which p can we still guarantee that G_p contains H?

G_p = random subgraph of G where each edge is retained with probability p.

Perturbed Random Graph: Let G be a graph with minimum degree $\delta \geq \alpha n$.

Let $G' = G \cup G(n,p)$. For which p can we guarantee that G' contains H?

Eg: H = Hamiltonian cycle.

- $p_c(\mathcal{H}) = \Theta\left(\frac{\log n}{n}\right)$. (Pósa)
- (Dirac) If G has minimum degree $\geq \frac{n}{2}$ then G has a Hamiltonian cycle (tight).

(Krivelevich–Lee–Sudakov)

- Given G with minimum degree $\geq \frac{n}{2}$, $G_{C\log n/n}$ has a Ham. cycle with high probability.
- For $\alpha \in (0, \frac{1}{2})$: If G has minimum degree $\delta \geq \alpha n$, then for $p = C/n$, $G' = G \cup G(n,p)$ contains a Ham. cycle. (Bohman–Frieze–Martin 03)
- If $\delta(G) \geq \frac{n}{2}$, then G contains at least $(cn)^n$ Hamiltonian cycles

(Sárközy–Selkow–Szemerédi, constant by Cuckler and Kahn)

Extremal Combinatorics

Thm (Corrádi – Hajnal): If $\delta(G) \geq \frac{2}{3}n$, then G contains a triangle factor.

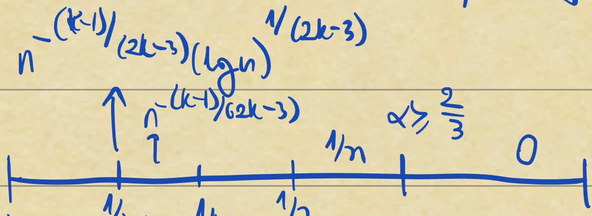
Thm (Hajnal – Szemerédi): If $\delta(G) \geq \frac{k}{k+1}n$, then G contains a K_{k+1}-factor.

Dirac threshold $\delta_{\ell,k} = \lim_{n\to\infty} \frac{t(n,k,\ell)}{\binom{n}{k-\ell}}$, $t(n,k,\ell) =$ min d s.t. $\delta_\ell(H) \geq d \Rightarrow$ perfect matching.

Thm (Komlós – Sárközy – Szemerédi): If $\delta(G) \geq \left(\frac{1}{2}+\varepsilon\right)n$, then G contains a bounded degree spanning tree T.

Thm (KSS): If $\delta(G) \geq \left(1-\frac{1}{k+1}\right)n$, then G contains a k^{th} power of a Hamiltonian cycle.

Perturbed Random Graph, Square of Ham. cycle



Probabilistic Combinatorics

Robust version (ABCDJMRS)

If $\delta(G) \geq \frac{2}{3}n$, then $G_{C(\log n)^{1/3} n^{-2/3}}$ contains a triangle factor.

(P.-Sah-Sawhney-Simkin): $G_{C(\log n)^{1/\binom{k+1}{2}} n^{-2/(k+1)}}$ contains a K_{k+1}-factor.

(PSSS): If $\delta(G) \geq \frac{k}{k+1}n$, then G contains $(cn)^{kn/(k+1)}$ K_{k+1}-factors.

(PSSS, KKKOP): If $\delta_\ell(H) \geq (\delta_{\ell,k}+\varepsilon)\binom{n}{k-\ell}$ then $H_{C(\log n)n^{1-k}}$ contains a perfect matching.

(PSSS): If $\delta(G) \geq \left(\frac{1}{2}+\varepsilon\right)n$, then $G_{C(\log n)/n}$ contains a copy of T.

If $\delta(G) \geq \left(1-\frac{1}{k+1}\right)n$, then $G_{Cn^{-1/k}}$ contains a k^{th}-power of Ham cycle $(k \geq 2)$. (KNP for $k=2$)

Thm (Nenadov-P. 2(?)): For $k \geq 3$, $\delta(G) \geq \left(\frac{1}{k+1}+\varepsilon\right)n$, then $G \cup G(n,p)$ contains a k^{th} power of the cycle for

$1/\sqrt[d]{n}$ $\quad 1/(k+1) \quad 1/k \quad 1/2 \quad 1$ | $p \geq Cn^{-1/(k-1)}$.

(Antoniuk, Dudek, Reiher, Ruciński, Schacht; Böttcher, Parczyk, Sgueglia, Skokan)

Regularity Method:

Def: $G = (A, B, E)$ is ε-regular if $\forall A' \subseteq A, B' \subseteq B, |A'| \geq \varepsilon |A|, |B'| \geq \varepsilon |B|$:

$$|d(A', B') - d(A, B)| \leq \varepsilon.$$

Here $d(A', B') = \dfrac{|E(A', B')|}{|A'||B'|}$.

Thm (Szemerédi's Regularity Lemma): $\forall \varepsilon > 0, \exists M(\varepsilon)$ s.t. the following holds. For any graph G, there is an equitable partition of $V(G)$ into at most $M(\varepsilon)$ parts $V_1, \dots, V_m$ such that for all but at most εm^2 pairs of parts V_i, V_j, $G[V_i, V_j]$ is ε-regular.

Def: G $= (A, B, E)$, $|A| = |B| = n$ is ε-super-regular if G is ε-regular with density d, and every vertex has degree $(d \pm \varepsilon)n$.

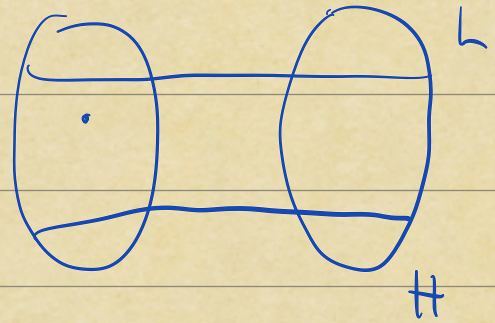
We say G is (ε, δ)-super-regular if G is ε-regular, and every vertex has degree at least δn.

Lem: For a bipartite graph $G = (A, B, E)$ which is (ε, δ)-super-regular, for any $\bar{d} \leq \delta - C\varepsilon$, G contains a spanning subgraph which is ε'-super-regular with density $\bar{d}$ where $\varepsilon' = O(\varepsilon^{1/2})$.

Pf: Let d_0 be the density of G. By ε-regularity, all but $O(\varepsilon n)$ vtx have degree $(d_0 \pm \varepsilon)n$.

Let $\hat{G}$ be obtained by keeping each edge with prob $\frac{d}{d_0}$, $d = \bar{d} + C\varepsilon$,

let L, H be the vtx with degree less than $(d-2\varepsilon)n$ & higher than $(d+2\varepsilon)n$.



For every $v \in L$, we add $dn - \deg_{\hat{G}}(v)$ edges between v and $(L \cup H)^c$.

For $v \in H$, remove $\deg_{\hat{G}}(v) - dn$ edges between v and $(L \cup H)^c$.

Now every vtx has degree $(d \pm O(\varepsilon))n$, and the density $\geq \bar{d}$.

Finally, obtain $\bar{G}$ by removing $\leq O(\varepsilon n)$ edges around each vtx s.t. edge density is exactly $\bar{d}$.

The Blow-up Lemma

Thm (Komlós - Sárközy - Szemerédi 97): $\forall r, \Delta, \delta > 0$, $\exists \varepsilon > 0$ s.t.

Let R be a graph on r vtx. Consider a graph G on $V_1 \cup \dots \cup V_r$, $|V_i| = n$, s.t. (V_i, V_j) is (ε, δ)-super-regular $\forall \{i,j\} \in E(R)$.

Let H be a graph with a homomorphism $h: H \to R$ and $|h^{-1}(i)| \leq n$ $\forall i$. Assume $\Delta(H) \leq \Delta$. Then G contains a copy of H.

Random Blow-up Lemma:

Def: Let λ be a distribution on $\phi : X \to Y$. We say λ is p-vertex-spread if $\forall x_1, \dots, x_k, y_1, \dots, y_k$:

$$\mathbb{P}_{\phi \sim \lambda}(\phi(x_i) = y_i \ \forall i) \leq p^k.$$

Thm (Nenadov - P. 24): $\forall r, \Delta, \delta, \alpha > 0, \exists \varepsilon, \beta > 0$ s.t.

Let R be a graph on r vtx. Consider a graph G on $V_1 \cup \dots \cup V_r$, $|V_i| = n$, s.t. (V_i, V_j) is (ε, δ)-super-regular $\forall \{i,j\} \in E(R)$.

Let H be a graph with a homomorphism $h : H \to R$ and $|h^{-1}(i)| \leq n \ \forall i$.
Assume $\Delta(H) \leq \Delta$.
Let $W \subseteq V(H)$, $|W| \leq \beta n$ be given. For each $x \in W$, given $W_x \subseteq V_{h(x)}$ of size $\geq \alpha n$.
Then there exists an $O(1/n)$-vertex-spread distribution λ on embeddings $\phi : H \to G$ s.t. $\phi(x) \in W_x \ \forall x \in W$.

Sketch of the Random Blow-up Lemma:

Key tool:

Lem (PSSS): $\forall d_0, \delta > 0$, $\exists \varepsilon > 0$: G (ε,δ)-super-regular then $\exists\ \mu$ on perfect matchings in G which is $O_\delta(1/n)$-spread.

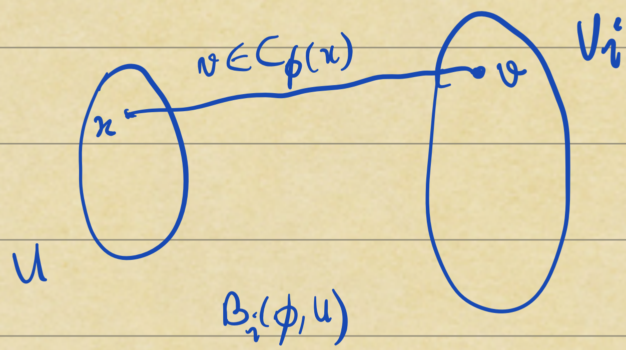
Let $H_i = h^{-1}(i)$. Choose $B_i, D_i \subseteq H_i \setminus W$: $|B_i| = \delta_0 n$, $|D_i| = \beta n$, $B \cup D$ ind set $(B = \bigcup B_i)$, every 2 vtx of $B \cup D$ have distance ≥ 4.

Quasirandom embedding:

Def: For $S \subseteq V(H)$, $\phi : H[S] \hookrightarrow G$ quasirandom if

- $\forall x \in V(H) \setminus S$, $$C_\phi(x) = W_x \cap \Big(\bigcap_{y \in N_H(x) \cap S} N_G(\phi(y)) \Big)$$ has $$|C_\phi(x)| \geq (d-\varepsilon)^{|N_H(x) \cap S|} |W_x|.$$
- for each i, at most $\varepsilon' |S| n$ pairs $(x,y) \in (H_i \setminus (N_H(D) \cup S))^2$ has $$|C_\phi(x) \cap C_\phi(y)| \leq (d+\varepsilon)^{|N_H(x) \cap S| + |N_H(y) \cap S|} n.$$

Lem: ϕ S-quasirandom. For $i \in R$, $U \subseteq H_i \setminus S$:



If $U \cap (W \cup N_H(D)) = \phi$,

$|U| \geq \delta_3 n$,

$|N_H(x) \cap S| = \ell$ $\forall x \in U$, then

$B_i(\phi, U)$ is ε''-regular, dens $\geq (d-\varepsilon)^\ell$.

[Quasirandomness $\Leftarrow$ $\sum_{x,y} |C_\phi(x) \cap C_{\phi}(y)|^2$ small].

Embedding: Order $V(H)$ s.t. $N_H(B)$ comes first, B comes last.

Embed $\{x_1, x_2, \dots, x_j\} = X_j$.

Phase 1: For $j \geq 0$,

- def $A_{j+1} \subseteq C_\phi(x_{j+1}) \setminus \phi(X_j)$ the set of v s.t.
  - $|N_G(v) \cap (C_\phi(y) \setminus \phi(X_j))| \geq (d-\varepsilon)\, |C_\phi(y) \setminus \phi(X_j)|$ $\forall\, y \in N_H(x_{j+1}) \setminus X_j$.
  - Extending ϕ to $\phi(x_{j+1}) = v$ gives a quasirandom embedding.

Pick $\phi(x_{j+1})$ from A_{j+1} unif at random.

- If $j = s$, $s = \delta_2 n$, def $L_j = \{v \notin X_j : |C_\phi(v) \setminus \phi(X_j)| < \delta_1 n\}$,

move L_j forward (may include vtx of B).

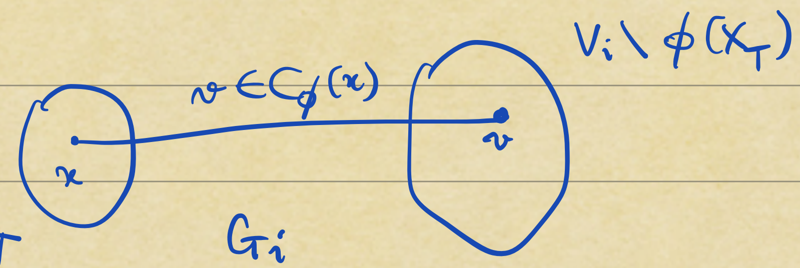
All of $N_H(B)$ already embedded

(Exceptional step) For $j = |N_H(B)|$, $E_i = \{v : |\{b \in B_i,\ v \in C_\phi(b)\}| < \delta_1 |B_i|\}$

Take random injection $\rho_i : E_i \to D_i$, move $\bigcup \rho_i(E_i)$ to beginning of ordering. For $x \in \rho_i(E_i)$, set $\phi(x) = \rho_i^{-1}(x)$.

Let T be the # vtx embedded.

Phase 2: For each i, consider



Sample $O(1/n)$-spread perfect matchings in $G_i \longrightarrow$ Final embedding ϕ.

Obs: From quasirandomness of embedding, $L_j = \phi$ for $j < |N_H(B)|$, so first embed all of $N_H(B)$.

By Lem, $B_i(\phi, B_i)$ ε''-regular, density $\geq (d-\varepsilon)^{\Delta} \Rightarrow |E_i| \leq \varepsilon'' n < |D_i|$

Claim: For $j \vdots s$, $|L_j \setminus (W \cup N_H(D))| < r(\Delta+1)\delta_3 n$:

Choose l st. $U_i^l \subseteq (H_i \cap L_j) \setminus (W \cup N_H(D))$ vtx x s.t. $|N_H(x) \cap X_j| = l$ has size $\geq \delta_3 n$.

Lem $\Rightarrow B_i(\phi, U_i^l)$ ε''-regular.

$\#$ vtx moved up $\leq O(\delta_3/\delta_2)n \ll \delta_0 n$ so $\geq \frac{\delta_0 n}{2}$ vtx of B unembedded. $\Rightarrow \exists x \in U_i^l$ with $> ((d-\varepsilon)^l - \varepsilon'')n$ neighbors in unembedded vtx. $\square$

$\Rightarrow$ Vtx moved up are always embedded before the next multiple of s phase.

Claim: $\forall j$, $x \in V(H) \setminus X_j$, $|C_\phi(x) \setminus \phi(X_j)| \geq (d-\varepsilon)^{\Delta+1} \delta_1 n - 2s$.

If $x \notin L_j$, $|C_\phi(x) \setminus \phi(X_{j+s})| \geq (d-\varepsilon)^{d_1} \delta_1 n - s$,

$d_1 = |N_H(x) \cap \{x_{j+1}, \dots, x_{j+s}\}|$.

If x embedded $\rightarrow \checkmark$, else consider $x \in L_{j+s}$.

$|C_\phi(x) \setminus \phi(X_{j+2s})| \geq (d-\varepsilon)^{d_2}((d-\varepsilon)^{d_1}\delta_1 n - s) - s$,

$d_2 = |N_H(x) \cap \{x_{j+s+1}, \dots, x_{j+2s}\}|$.

x will be embedded before step $j+2s$. $\square$

Claim: $|A_{j+1}^i| \gg \delta_2 n$.

By regularity, all but $O(\varepsilon)n$ $v \in W_{x_{j+1}^i}$ has

$$|N_G(v) \cap C_\phi(y)| \geq (d-\varepsilon)\, |C_\phi(y)|,$$

$$|N_G(v) \cap (C_\phi(y) \setminus \phi(X_j))| \geq (d-\varepsilon)\, |C_\phi(y) \setminus \phi(X_j)| \qquad \forall y \in N_H(x_{j+1}) \setminus X_j.$$

Let $P = \{ (x,y) \in (H_i \setminus (N_H(S) \cup S))^2 : \overset{(*)}{|C_\phi(x) \cap C_\phi(y)| \leq (d+\varepsilon)^{|N_H(x) \cap S| + |N_H(y) \cap S|}\, n},$
x or $y \in N_H(x_{j+1}) \}$.

For random $v \in V_{h(x_{j+1})}$, $\mathbb{E}[\#\text{pairs failing (2) after } \phi(x_{j+1}) = v] \leq \varepsilon P$.

W.p. $1-\varepsilon'$, at most $\varepsilon' n$ new pairs fail $(*)$.

$\Rightarrow$ All but $O(\varepsilon' n)$ vtx of $C_\phi(x_{j+1}) \setminus X_j$ belongs to A_{j+1}^i. $\square$

Phase 2: $|B_i \setminus X_T| \geq |B_i| - O(\delta_3/\delta_2)n$,

By Lem, $B_i(\phi, B_i \setminus \phi(X_T))$ ε''-regular.

$P_i = V_i \setminus \phi(X_T)$: From exceptional step, every $v \in F_i$ has deg $\geq \delta_4 |B_i| - O(\delta_3 n/\delta_2)$,

while Claim $\Rightarrow$ $|C_\phi(b) \setminus \phi(X_T)| > \delta_2 n$.

$\Rightarrow \exists$ spread perfect matching.