An upper bound on p_E:

$$p_E = \sup_p \left\{ \begin{array}{ll} \min\limits_{g : 2^X \to \{0,1\}} & \sum\limits_{G \in 2^X} g(G)\, p^{|G|} \\ \text{s.t.} & \sum\limits_{G \subseteq H} g(G) \geq 1 \quad \forall H \in \mathcal{H} \end{array} \leq \frac{1}{2} \right\}$$

$$\leq \sup_p \left\{ \begin{array}{ll} \min\limits_{w : 2^X \to [0,1]} & \sum\limits_{G \in 2^X} w(G)\, p^{|G|} \\ \text{s.t.} & \sum\limits_{G \subseteq H} w(G) \geq 1 \quad \forall H \in \mathcal{H} \end{array} \leq \frac{1}{2} \right\}$$

fractional cover w

$$= \sup_p \left\{ \begin{array}{ll} \max\limits_{\lambda : \mathcal{H} \to \mathbb{R}_{\geq 0}} & \sum \lambda(H) \\ \text{s.t.} & \sum\limits_{H \supseteq G} \lambda(H) \leq p^{|G|} \quad \forall G \in 2^X \end{array} \leq \frac{1}{2} \right\}$$

$$= \inf_p \left\{ \begin{array}{l} \exists\, \lambda : \mathcal{H} \to \mathbb{R}_{\geq 0}, \quad \sum \lambda(H) = 1, \\ \sum\limits_{H \supseteq G} \lambda(H) \leq 2p^{|G|} \quad \forall G \in 2^X \end{array} \right\}.$$

Def (Talagrand): Fractional expectation threshold

$$p_f(\mathcal{H}) = \inf_p \left\{ \begin{array}{l} \exists\, \lambda : \mathcal{H} \to \mathbb{R}_{\geq 0}, \quad \sum \lambda(H) = 1, \\ \sum\limits_{H \supseteq G} \lambda(H) \leq 2p^{|G|} \quad \forall G \in 2^X \end{array} \right\}$$

p-spread measure λ

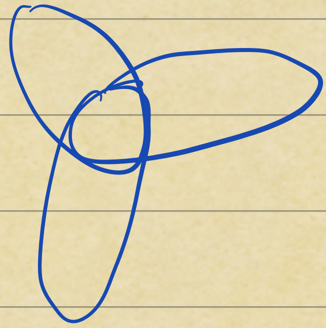
We say $\mathcal{H}$ is p-spread if $p_f(\mathcal{H}) \leq p$ (so p-spread $\Rightarrow$ not p-small).

E.g.:

- Verify that the uniform distribution on r-uniform perfect matchings is $O(n^{1-r})$-spread
- For spanning tree T with max degree Δ, verify that the uniform distribution is $O_\Delta(1/n)$-spread.

Connections to Set Systems, Sunflowers:

r-sunflower:



$A_1, \ldots, A_r$: $A_i \cap A_j = \bigcap_{k=1}^{r} A_k \;\; \forall i \neq j$

Sunflower Conjecture : Any C_r^k k-sets have an r-sunflower.

Let $\mathcal{H}$ be an arbitrary family of h^k k-sets. Let μ be the uniform distribution on $\mathcal{H}$. Then if $\exists S$ s.t. $\mu(\langle S \rangle) > 2p^{|S|}$, $p = \frac{1}{h}$,

we replace $\mathcal{H}$ by $\{H \setminus S : H \in \mathcal{H},\ S \subseteq H\} \rightarrow (k-|S|)$-uniform,

$$\geq h^k \, 2\left(\frac{1}{h}\right)^{|S|} > h^{k-|S|} \text{ sets}$$

Spread lemma

Thm (Alweiss - Lovett - Wu - Zhang '19) : $\mathcal{H}$ $\frac{c}{\lg l \lg\lg l}$-spread, l-bounded.

Then $X_{c'} \in \langle \mathcal{H} \rangle$ with probability $\geq \frac{1}{2}$.

Using the argument of ALWZ breakthrough, FKNP proved

Thm (Frankston - Kahn - Narayanan - Park '19) : $\mathcal{H}$ p-spread, l-bounded,

$X_{Cp \lg l} \in \langle \mathcal{H} \rangle$ with probability $\geq \frac{1}{2}$.

$\mathcal{H}$ p-spread $\Rightarrow$ $\mathcal{H}$ not p-small, so direct consequence of Kahn - Kalai.

Under p-spread property: Same proof of Key Lemma gives:

Lem: Let $\mathcal{H}$ be p-spread, ℓ-bounded. Then with prob $\frac{1}{2}$ over $W \sim X_{Cp}$, a $(1-2(4/C)^{\ell/2})$-fraction of $H \in \mathcal{H}$ have $|T(W,H)| \leq \ell/2$.

$\hookrightarrow$ Directly iterate.

$$\mathbb{E}_{W \sim X_{Cp}}\left[\#\{H : |T(W,H)| > \ell/2\}\right] \leq \sum_{Z \in \langle \mathcal{H} \rangle} P_{Cp}(Z) \sum_{t > \ell/2} \frac{(1-Cp)^t}{(Cp)^t} \cdot 2^\ell p^t |\mathcal{H}|$$

Thm (ALWZ, FKNP, Rao, Bell–Chueluecha–Warnke): A family of $(Cr\log \ell)^\ell$ ℓ-sets contains an r-sunflower.

Producing the sunflower: Take random partition of X to $2r$ sets.

$\mathcal{H}$ is $\frac{1}{Cr\log \ell}$-spread $\Rightarrow$ $\mathbb{E}$ #parts in $\langle \mathcal{H} \rangle \geq r$.

"Smooth" spread property: $(\mathcal{H}, \lambda)$ $(p; r_1, \dots, r_k)$-spread if $\mathcal{H}$ is r_1-bounded, $r_k = 1$

and for $|A| \leq r_i$ and $t \geq r_{i+1}$,

$$\lambda(\{H \in \mathcal{H} : |H \cap A| = t\}) \leq p^t.$$

Thm (Spiro 23, Espuny Díaz–Person 23): If $\mathcal{H}$ admits a $(p; r_1, \dots, r_k)$-spread measure, then $p_c(\langle \mathcal{H} \rangle) \leq Cpk$.

Pf: Wlog, λ uniform on $\mathcal{H}$.

For $W \sim X_{Cp}$,

$$\mathbb{E}_{W \sim X_{Cp}}\left[\#\{H : |T(W,H)| \geq r_2\}\right]$$

$$\leq \sum_{Z \in \langle \mathcal{H} \rangle} P_{Cp}(Z) \sum_{t \geq r_2} \frac{(1-Cp)^t}{(Cp)^t} \sum_{\substack{T \subseteq \hat{H}(Z) \\ |T| = t}} \#\{H : T \subseteq H\}$$

$$\leq \sum_{Z \in \langle \mathcal{H} \rangle} P_{Cp}(Z) \sum_{t \geq r_2} \frac{(1-Cp)^t}{(Cp)^t} \; p^t \; |\mathcal{H}|$$

So with high probability, a $(1 - 2(1/C)^{r_2})$ - faction of $H \in \mathcal{H}$ have $|T(W, H)| \leq r_2$.

Conditioned on succes, repeat argument with $\mathcal{H}_1 = \left\{ T(W,H) : |T(W,H)| \leq r_2 \right\}$

Claim: The k^{th} powers of the Hamiltonian cycle is

$(n^{1-k}; r_1, \dots, r_t)$ - spread for $r_1 = kn$, $r_{i+1} = r_i n^{-\gamma_k}$, for some $\gamma_k > 0$.

Sunflowers in Set systems with bounded VC-dimension:

$\mathcal{H}$ has VC-dim $\leq d$ if no $(d+1)$-element set S has the property that

$$\forall T \subseteq S, \; \exists H \in \mathcal{H} : H \cap S = T.$$

Thm: $\mathcal{H}$ ℓ-bounded, VC-dim $\leq d$, and $|\mathcal{H}| \geq (Cr(\log d + \log_* \ell))^\ell$ then $\mathcal{H}$ contains an r-sunflower.

Pf: Standard greedy procedure $\longrightarrow$ $\mathcal{H}$ is $\dfrac{1}{Cr(\log d+\log_* \ell)}$ - spread, VC-dim $\le d$.

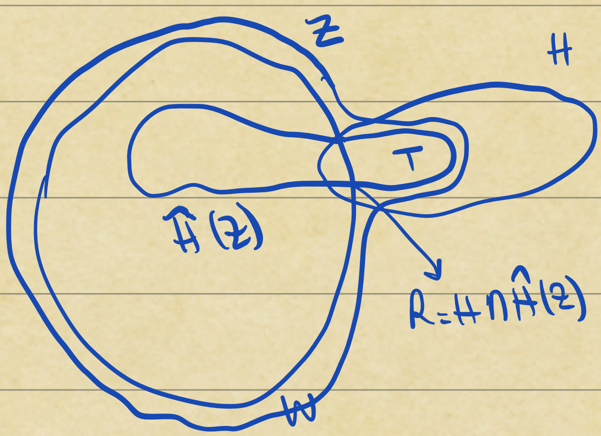
Thm (Bologna - Bernshteyn - Delcourt - Ferber -) p. 24: $\mathcal{H}$ ℓ-bounded, VC-dim $\le d$, not p-small.

Then $X_{Cp(\log d+\log_* \ell)} \in \langle \mathcal{H} \rangle$ with prob $1/2$.

Pf: We show that for $\mathcal{U}_W = \{ T(W,H) : |T(W,H)| \geq Cd \log_2 \ell \}$,

$$\overline{\mathcal{U}}_W = \{ H \cap \hat{A}(Z) : Z = W \cup T(W,H),\ |T(W,H)| \geq Cd \log \ell \}$$

$$\mathbb{E}_{W \sim X_{Cp}} [c_p(\overline{\mathcal{U}}_W)] \leq \ell^{-d}. \qquad (*)$$



$$\mathbb{E}_{W \sim X_{Cp}} [c_p(\overline{\mathcal{U}}_W)]$$

$$\leq \sum_{Z \in \langle \mathcal{H} \rangle} P_{Cp}(Z) \sum_{t \geq Cd \log \ell} \frac{(1-Cp)^t}{(Cp)^t} \cdot$$

$$\sum_{u \geq t} p^u \, \#\{ U \subseteq \hat{A}(Z),\ |U| = u,\ U = H \cap \hat{A}(Z) \} \cdot \binom{u}{t}$$

Sauer - Shelah; Perles, Vapnis - Chervonenkis:

$$\#\{ U \subseteq \hat{H} :\ |U| = u,\ U = H \cap \hat{H} \} \leq \sum_{j=1}^{d} \binom{|\hat{H}|}{j} \leq \left(\frac{e\ell}{d} \right)^d.$$

Sharp Selector Process and Rounding fractional covers:

Thm (P., 2024) Let $\mathcal{H}$ be not p-small. For any valuation $\lambda_H : H \to \mathbb{R}_{\geq 0}$ with $\sum_{x \in H} \lambda_H(x) \geq 1 \ \forall H$, then with probability at least $\frac{1}{3}$, for $q = Cp \log \frac{1}{\varepsilon}$

$$\max_{H \in \mathcal{H}} \sum_{x \in H \cap X_q} \lambda_H(x) \geq 1 - \varepsilon.$$

Conjecture (Talagrand): $p_f(\mathcal{H}) \leq C p_E(\mathcal{H})$, i.e. if $\mathcal{H}$ admits a fractional cover w with $\sum_W w(W) p^{|W|} < \frac{1}{2}$, then $\mathcal{H}$ has a cover $\mathcal{G}$ with $\sum_{G \in \mathcal{G}} (p/C)^{|G|} \leq \frac{1}{2}$.

("Rounding fractional cover")

Thm (P., 24): If $\mathcal{H}$ admits a fractional cover w supported on sets of size at most t with $\sum_W w(W) p^{|W|} \leq \frac{1}{2}$, then $\mathcal{H}$ has a cover $\mathcal{G}$ with

$$\sum_{G \in \mathcal{G}} (p/C \log t)^{|G|} \leq \frac{1}{2},$$

Pf: Set $\lambda_H(x) = \dfrac{\sum_{W \subseteq H, x \in W} w(W)/|W|}{\sum_{W \subseteq H} w(W)}$

If $\mathcal{H}$ is not $\frac{cp}{\log t}$-small, then with probability $1/3$,

$$\max_{H} \sum_{x \in X_{c'p} \cap H} \lambda_H(x) \geq 1 - \frac{1}{2t}.$$

- $$\sum_{x \in X_{c'p} \cap H} \lambda_H(x) = \frac{\sum_{W \subseteq H} w(W) - \sum_{W \subseteq H} w(W) \frac{|W \setminus X_{c'p}|}{|W|}}{\sum_{W \subseteq H} w(W)}$$

$$\leq 1 - \frac{1}{t} \frac{\sum_{W \subseteq H} w(W)\, \mathbb{1}(W \not\subseteq X_{c'p})}{\sum_{W \subseteq H} w(W)}$$

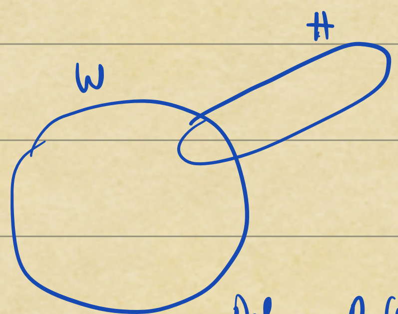
$$\Longrightarrow \quad \max_{H} \sum_{W \subseteq H \cap X_{c'p}} w(W) \geq \frac{1}{2}.$$

But $$\mathbb{E}\left[\sum_{W \subseteq X_{c'p}} w(W)\right] = \sum w(W)\,(c'p)^{|W|} < \frac{1}{6},$$

Contradiction. □

Ideas for proving (Sharp) Selector Process Conjecture:

Def:



Write $H = (h_1, \ldots, h_\ell)$, $\lambda_H(h_1) \geq \cdots \geq \lambda_H(h_\ell)$.

Let $b = b(W,H)$ be smallest s.t.

$$\sum_{j > b} \lambda_H(h_j)\, \mathbb{1}(h_j \in W) \geq c \sum_{j > b} \lambda_H(h_j).$$

Def $R(W,H) = \{h_1, \ldots, h_b\}$.

Z is t-feasible if $\exists\, W \subseteq Z,\ H \in \mathcal{H}$: $|W| = |Z| - t$, $Z \supseteq R(W,H)$.

Minimum fragment $T(W,H)$ is the minimal t smallest subset of H s.t. $W \cup T(W,H)$ feasible.

Obs: If $\max_{H} \sum_{x \in H \cap W} \lambda_H(x) < c$ then $T(W,H) \neq \phi$ $\forall H$.

For any $(\hat{W}, \hat{H})$ witnessing feasibility of $Z = W \cup T(W,H)$,

$T(W,H) \subseteq R(\hat{W}, \hat{H})$, and $|T(W,H)| \geq c\,|R(\hat{W}, \hat{H})|$.

For the sharp version: Given a sequence $W_1, \ldots, W_s$, define $b_1 = b(W_1, H)$,

$R_1 = R(W_1, H)$. Then let $H_1 = H \setminus R_1$, $b_2 = b(W_2, H_1)$, $R_2 = R(W_2, H_1)$, ...

$(Z_1, \ldots, Z_s)$

is $(t_1, \ldots, t_s)$-feasible if $\exists_{H}\, W_1, \ldots, W_s$: $W_i \subseteq Z_i$, $|W_i| = |Z_i| - t_i$,

$Z_i \supseteq R_i(W_i, H)$.

Minimum fragment $(T_1, \ldots, T_s)$ minimal (lexico) subsets of H s.t.

$(W_1 \cup T_1, \ldots, W_s \cup T_s)$ feasible

* The selector process conjecture extended to some other processes + proof simplification

by Bednorz- Martynek- Meller.