Talagrand's Selector Process Conjecture:

There is $C>0$ s.t.: If $\mathcal{H}$ is not p-small, for any valuation $\lambda_H : H \to \mathbb{R}_{\geq 0}$ with $\sum_{x \in H} \lambda_H(x) \geq 1 \quad \forall H \in \mathcal{H}$,

$$\mathbb{E}\left[\max_{H \in \mathcal{H}} \sum_{x \in H \cap X_p} \lambda_H(x)\right] \geq \frac{1}{C}.$$

Cor: For $\mathcal{H}$ l-uniform, not p-small, with probability at least $1/2$,

$$\max_{H \in \mathcal{H}} |X_{Cp} \cap H| \geq cl.$$

Connections:

- Linear program under random sparsification.
  Optimization

- Selector process:

  For iid $\mathrm{Ber}(p)$ Y_x, define $Z_\lambda(Y) = \sum_x \lambda(x) Y_x$,

  and $Z_\Lambda = \sup_{\lambda \in \Lambda} Z_\lambda$.

Conjecture (Talagrand, 2006): For Λ consisting of nonnegative vectors λ,

$$\mathcal{E} = \{ Z_\Lambda \geq C\, \mathbb{E} Z_\Lambda \} \text{ is } p\text{-small.}$$

- Analogous to results for Gaussian process

· Key remaining component to suprema of general empirical processes.

(Talagrand, "Unfulfilled dreams")

The selector process conjecture (and the version for empirical processes) proven by Park - P. (24).

Sharp version of Talagrand's Selector Process Conjecture:

Thm (P., 2024) Let $\mathcal{H}$ be not p-small. For any valuation $\lambda_H : H \to \mathbb{R}_{\geq 0}$ with $\sum_{x \in H} \lambda_H(x) \geq 1 \ \forall H$, then with probability at least $\frac{1}{3}$, for $q = Cp \log \frac{1}{\varepsilon}$

$$\max_{H \in \mathcal{H}} \sum_{x \in H \cap X_q} \lambda_H(x) \geq 1 - \varepsilon.$$

Implies directly the Kahn - Kalai conjecture:

For KK: set $\lambda_H(x) = \frac{1}{|H|}$ for $x \in H$, and pick $\varepsilon = \frac{1}{2\ell}$ $\left(\ell = \max_{H \in \mathcal{H}} |H|\right)$.

(Sharp) Selector Process Conjecture: Significant weighted generalization.

Cor: Change from requiring the whole (spanning) structure in the Kahn - Kalai Conj to almost all replaces $\log \ell$ by $\log \frac{1}{\varepsilon}$.

E.g. For ℓ-uniform H, with probability $\geq \frac{1}{2}$, $\max |X_{Cp} \cap H| \geq .99\ell$.

* Much more involved proof, but the challenge and rich picture in the weighted setup of Talagrand's conjecture motivates the proof of the Kahn-Kalai conjecture.

Proof of the Kahn–Kalai Conjecture:

Key def: The minimum fragment $T = T(W, H)$ is the minimum subset of H with $W \cup T \in \langle \mathcal{H} \rangle$.

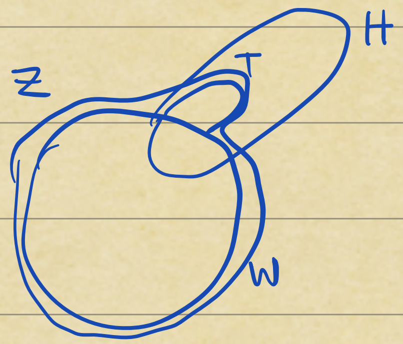
Alternately: Z feasible if $Z \in \langle \mathcal{H} \rangle$, T minimum s.t. $W \cup T$ feasible.

Key Lemma: Let $\mathcal{H}$ be ℓ-bounded $\left(\max_{H \in \mathcal{H}} |H| \leq \ell\right)$. Let $\mathcal{U}_W = \left\{ T = T(W,H) : |T| > \ell/2 \right\}$

Then $\mathbb{E}_{W \sim X_{Cp}}\left[C_p(\mathcal{U}_W) \right] \leq 2\,(4/C)^{\ell/2}$.

Proof of Key Lemma

Key Property:



Given $Z = W \cup T$, $T = T(W, H)$,
for all $H' \in \mathcal{H}$ s.t. $H' \subseteq Z$,
$T \subseteq H'$.

(If not, $T \cap H'$ gives a smaller fragment).

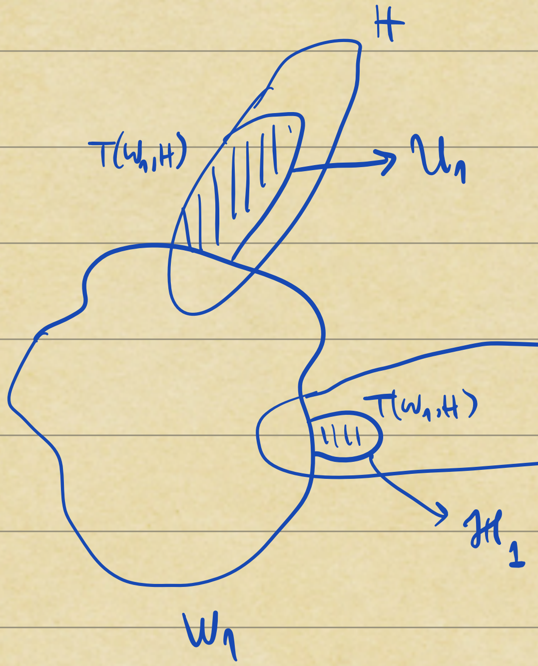
By the key Property: Choose for every $Z \in \langle \mathcal{H} \rangle$ an arbitrary $\hat{H}(Z) \subseteq Z$, $\hat{H}(Z) \in \mathcal{H}$.
then
$$\left\{ (Z \setminus T, T) : \begin{matrix} T \subseteq \hat{H}(Z) \\ |T| > \ell_2 \end{matrix} \right\} \supseteq \left\{ (W, T) : T = T(W, H),\ |T| > \ell/2 \right\}.$$

Let $P_{cp}(Y) = (cp)^{|Y|}(1-cp)^{|X \setminus Y|}$:
$$\mathbb{E}_{W \sim X_{cp}}[\varphi(\mathcal{U}_W)] = \mathbb{E}_{W \sim X_{cp}}\left[\sum_T p^{|T|} \mathbb{1}\left(T = T(W,H) \text{ for some } H,\ |T| > \ell/2\right)\right]$$
$$= \sum_{(W,T)} P_{cp}(W)\, p^{|T|}\, \mathbb{1}\left(T = T(W,H) \text{ for some } H,\ |T| > \ell/2\right)$$
$$\leq \sum_{Z \in \langle \mathcal{H} \rangle} \sum_{\substack{T \subseteq \hat{H}(Z) \\ |T| > \ell_2}} P_{cp}(Z \setminus T)\, p^{|T|}$$
$$\leq \sum_{Z \in \langle \mathcal{H} \rangle} \sum_{t > \ell/2} \frac{P_{cp}(Z)\,(1-cp)^t}{(cp)^t} \cdot p^t \binom{\ell}{t}$$
$$\leq \sum_{Z \in \langle \mathcal{H} \rangle} P_{cp}(Z)\, 2\left(4/c\right)^{\ell/2} \leq 2\left(4/c\right)^{\ell/2}. \qquad \square$$

Proof of the Kahn-Kalai Conjecture: Sample $W_1, \dots \sim X_{cp}$.

Define $\mathcal{H} = \mathcal{H}_0, \mathcal{H}_1, \dots$: $\mathcal{H}_i$ $\ell 2^{-i}$-bounded,

$\mathcal{U}_1, \dots$: every $H \in \mathcal{H}_0$ contains a set in $\mathcal{H}_i$, or covered by $\mathcal{U}_1 \cup \dots \cup \mathcal{U}_i$.



$$\mathcal{U}_i = \{ T(W_i, H) : H \in \mathcal{H}_{i-1}, \ |T(W_i,H)| > \ell 2^{-i} \},$$

$$\mathcal{H}_i = \{ T(W_i, H) : H \in \mathcal{H}_{i-1}, \ |T(W_i,H)| \leq \ell 2^{-i} \}.$$

Obs: $\mathcal{H}_{C \log \ell} = \emptyset$ or $\{\emptyset\} \Rightarrow W_1 \cup \dots \cup W_{C\log \ell} \in \langle \mathcal{H} \rangle$.

$\Downarrow$

$\mathcal{H}$ covered by $\bigcup \mathcal{U}_i$

By key Lemma: $\mathbb{E}\, c_p(\mathcal{U}_i) \leq 2(4/C)^{\ell i/2}$, so check that

$\mathbb{E} \sum_i c_p(\mathcal{U}_i) \leq \frac{1}{4}$, but

$\mathcal{H}$ is not p-small $\Rightarrow$ $\mathbb{P}(\mathcal{H}_{C\log \ell} = \emptyset) \leq 2\mathbb{E} \sum c_p(\mathcal{U}_i) = \frac{1}{2}$

$\Rightarrow \mathbb{P}(W_1 \cup \dots \cup W_{C \log \ell} \in \langle \mathcal{H} \rangle) \geq \frac{1}{2}$. $\square$