Erdős-Rényi Random Graph $G(n,p)$: n vertices, each pair adjacent independently with probability p.

Motivating question: Given a graph H on n vertices, what is the probability that $G(n,p)$ contains H as a subgraph?

General setup: Finite set X.

X_p : random subset of X where each $x \in X$ is included independently with probability p

Eg: For $X = \binom{[n]}{2}$, $X_p \longleftrightarrow G(n,p)$.

A monotone property $\mathcal{P}$ on 2^X is a collection $\mathcal{P} \subseteq 2^X$ s.t.

$$S \in \mathcal{P},\ S \subseteq T \Rightarrow T \in \mathcal{P}.$$

Eg. $X = \binom{[n]}{2}$, $\mathcal{P} = \{\text{graphs containing } H \text{ as a subgraph}\}$.

Given $H \subseteq X$, write $\langle H \rangle = \{ W \subseteq X : H \subseteq W \}$,

for $\mathcal{H} \subseteq 2^X$, write $\langle \mathcal{H} \rangle = \bigcup_{H \in \mathcal{H}} \langle H \rangle$.

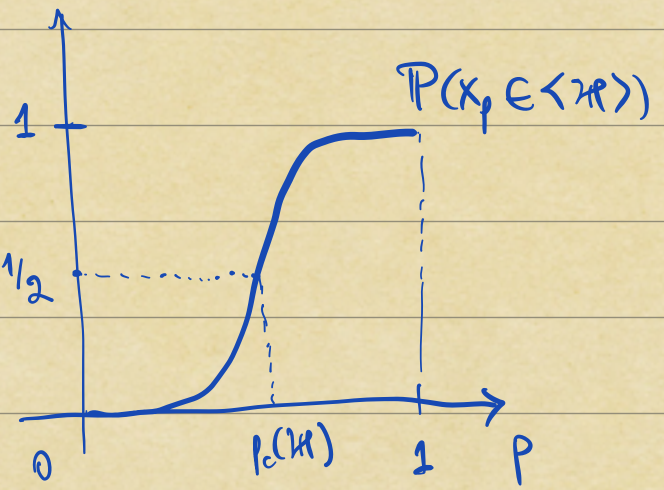
<u>Obs</u>: Every monotone property $\mathcal{P}$ can be written as $\langle \mathcal{H} \rangle$ for some $\mathcal{H} \subseteq 2^X$.

$\underline{Q}$: When does X_p typically satisfying $\mathcal{P}$, i.e. $X_p \in \langle \mathcal{H} \rangle$?

$\underline{\text{Def}}$: The threshold of the property $\mathcal{P}$, $p_c(\mathcal{H})$ is the value $p \in [0,1]$ s.t.

$$\mathbb{P}(X_p \in \langle \mathcal{H} \rangle) = 1/2.$$

Note: $p \mapsto \mathbb{P}(X_p \in \langle \mathcal{H} \rangle)$ is continuous, increasing from 0 to 1 unless $\langle \mathcal{H} \rangle = \phi$ or 2^X.



1) The first moment:

$\underline{\text{Def}}$: For $W \subseteq X$, let $N_{\mathcal{H}}(W) = \sum_{H \in \mathcal{H}} \mathbb{I}(H \subseteq W)$.

. $\{X_p \in \langle \mathcal{H} \rangle\} = \{N_{\mathcal{H}}(X_p) \geq 1\}$.

The first moment bound:

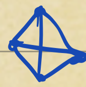
$$\mathbb{P}(X_p \in \langle \mathcal{H} \rangle) = \mathbb{E}[\mathbb{I}(N_{\mathcal{H}}(X_p) \geq 1)]$$

$$\leq \quad \mathbb{E}[N_{\mathcal{H}}(X_p)]$$

Linearity of expectation: $\quad \mathbb{E}[N_{\mathcal{H}}(X_p)] = \sum_{H \in \mathcal{H}} p^{|H|} := c_p(\mathcal{H})$

Thus: $\sum_{H \in \mathcal{H}} p^{|H|} \leq \frac{1}{2} \Rightarrow \mathbb{P}(X_p \in \langle \mathcal{H} \rangle) \leq \frac{1}{2}$, or $p_c(\mathcal{H}) \geq p$.

$\hookrightarrow$ First moment obstruction (naive)

Example: • $\mathcal{H} = \{$  K_4 on n vtx $\}$,

$$c_p(\mathcal{H}) = \binom{n}{4} p^6 \Rightarrow p_c(\mathcal{H}) = \Omega(n^{-2/3}).$$

• $\mathcal{H} = \{ K_{2\log n}$ on n vtx $\}$,

$$c_p(\mathcal{H}) = \binom{n}{2\log n} p^{\binom{2\log n}{2}}.$$

For $p = \frac{1}{2}$, $c_p(\mathcal{H}) \leq \exp(-\Omega(\log n \log\log n))$.

• H fixed graph, $\mathcal{H} = \{$copies of H on n vtx$\}$

$$c_p(\mathcal{H}) = \Theta\left(n^{v(H)} p^{e(H)}\right), \text{ so } p_c(\mathcal{H}) = \Omega\left(n^{-v(H)/e(H)}\right)$$

2) The second moment method:

When $\mathbb{E} N_{\mathcal{H}}(X_p)$ is small, $\mathbb{P}(X_p \in \langle \mathcal{H} \rangle)$ is small.

Is $\mathbb{P}(X_p \in \langle \mathcal{H} \rangle)$ large when $\mathbb{E} N_{\mathcal{H}}(X_p)$ is large?

If $N_{\mathcal{H}}(X_p)$ concentrates around $\mathbb{E}N_{\mathcal{H}}(X_p)$, then this is the case.

Second moment method: This happens when $\mathrm{Var}\, N_{\mathcal{H}}(X_p) = o\left((\mathbb{E}N_{\mathcal{H}}(X_p))^2\right)$.

Example: Fixed graph H.

$$\mathbb{E}[N_{\mathcal{H}}(X_p)^2] - (\mathbb{E}N_{\mathcal{H}}(X_p))^2 \le \sum_{H' \cap H'' \neq \emptyset} p^{|E(H') \cup E(H'')|}$$

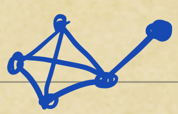
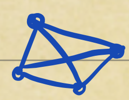
$$\le \sum_{H'} p^{2e(H)} \sum_{\emptyset \neq K \subseteq H'} p^{-e(K)} \sum_{H''} \mathbb{I}(H'' \supseteq K)$$

$$\le \sum_{\emptyset \neq K \subseteq H} p^{-e(K)} n^{v(H) - v(K)} \cdot p^{2e(H)} n^{v(H)}$$

$$= O_H\left((\mathbb{E}N_{\mathcal{H}}(X_p))^2\right) \max_{\emptyset \neq K \subseteq H} p^{-e(K)} n^{-v(K)}.$$

Thus, for $p \ge C n^{-\min_{\emptyset \neq K \subseteq H} \frac{v(K)}{e(K)}}$, then $\mathbb{P}(N_{\mathcal{H}}(X_p) \ge 1) \ge \frac{1}{2}$.

Conclude that $p_c(\mathcal{H}) = \Theta_H\left(n^{-\min \frac{v(K)}{e(K)}}\right)$, lower bound from first moment obstruction for K.

Eg: $H =$  , $K =$ 

For fixed size H, we also know sharper concentration:

Janson's Inequality:

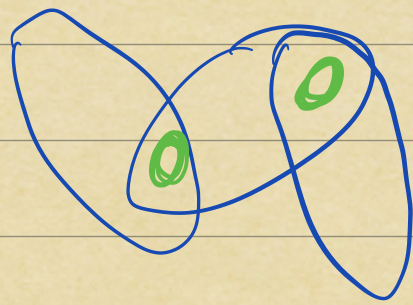
$$\mathbb{P}(N_{\mathcal{H}}(X_p) \le \mathbb{E}N_{\mathcal{H}}(X_p) - t) \le \exp(-t^2/2\bar{\Delta}).$$

$$\bar{\Delta} = \mathbb{E}N_{\mathcal{H}}(X_p) + \sum_{H' \cap H'' \neq \emptyset} p^{|E(H') \cup E(H'')|}.$$

3) The expectation threshold:

Even for fixed size H, we saw that the naive first moment obstruction can give the wrong prediction.

Def: For $\mathcal{H} \subseteq 2^X$, $\mathcal{G} \subseteq 2^X$ covers $\mathcal{H}$ if $\mathcal{H} \subseteq \langle \mathcal{G} \rangle$.



If $\mathcal{H}$ has cover $\mathcal{G}$ with $c_p(\mathcal{G}) = \sum_{G \in \mathcal{G}} p^{|G|} \leq \frac{1}{2}$, then

$$\mathbb{P}(X_p \in \langle \mathcal{H} \rangle) \leq \mathbb{P}(X_p \in \langle \mathcal{G} \rangle) \leq c_p(\mathcal{G}) \leq \frac{1}{2}.$$

We say $\mathcal{H}$ is p-small when there exists such cover $\mathcal{G}$.

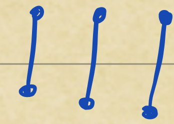
Def: The expectation threshold $p_E(\mathcal{H})$ is the smallest p s.t. $\mathcal{H}$ does not admit a cover $\mathcal{G}$ with $c_p(\mathcal{G}) \leq \frac{1}{2}$.

Claim: $$p_c(\mathcal{H}) \geqslant p_E(\mathcal{H}).$$

Before, we have shown that for fixed size H, $\mathcal{H} = \{\text{copies of } H\}$,

$$p_c(\mathcal{H}) = \Theta(p_E(\mathcal{H})).$$

Does this work for other $\mathcal{H}$?

Eg: $\mathcal{H} = \{\text{perfect matchings on } n \text{ vtx}\}$ (n even) 

$$\mathbb{E} N_{\mathcal{H}}(X_p) = \frac{n!}{2^{n/2}(n/2)!} \, p^{n/2} \leq (Cnp)^{n/2},$$

in fact $\quad p_E(\mathcal{H}) = \Theta\left(\frac{1}{n}\right)$: # perfect matchings containing $G \leq \dfrac{(n-2e(G))!}{2^{n/2-e(G)}(n/2-e(G))!}$.

But for any $\quad p \leq c \dfrac{\log n}{n}$: Even though $\mathbb{E} N_{\mathcal{H}}(X_p)$ huge,

isolated vtx in $G(n,p) \geq \Omega(n^c)$ whp (second moment)

$\Rightarrow N_{\mathcal{H}}(X_p) = 0$ whp.

(Pósa): $\quad p_c(\mathcal{H}) = \Theta(\log n / n)$.

The same $\log n$ gap for:
- Hamiltonian cycle
- Hypergraph perfect matching (Shamir's problem)
- Fixed spanning tree (bounded degree)
- ...

4) The Kahn-Kalai Conjecture:

Conjecture (Kahn - Kalai 2006): There exists $C>0$ s.t.

$$p_E(\mathcal{H}) \leq p_c(\mathcal{H}) \leq C\, p_E(\mathcal{H}) \log |X| ,$$

i.e. if $\mathcal{H}$ is not p-small, then

$$X_{Cp \log |X|} \in \langle \mathcal{H} \rangle \quad \text{with prob} \geq \frac{1}{2}.$$

Thm (Park - P.): $\exists C>0$ s.t. if $\mathcal{H}$ is not p-small, and $\max_{H \in \mathcal{H}} |H| \leq \ell$, then $X_{Cp \log \ell} \in \langle \mathcal{H} \rangle$ with high prob.

Applications/Eg:

- Hypergraph perfect matching:

$X = \binom{[n]}{r}$, $\mathcal{H} = \{r\text{-uniform perfect matchings on } n \text{ vtx}\}$

Check that
- $p_E(\mathcal{H}) \geq \dfrac{c}{n^{r-1}}$
- $p_E(\mathcal{H}) \leq \dfrac{C}{n^{r-1}}$

$\Rightarrow p_c(\mathcal{H}) \leq \dfrac{C \log n}{n^{r-1}}.$

- $p_c(\mathcal{H}) \geq \dfrac{c \log n}{n^{r-1}}$.

$$\Rightarrow p_c(\mathcal{H}) = \Theta\left(\frac{\log n}{n^{r-1}}\right).$$

(Earlier proven by Johansson - Kahn - Vu, 2008).

Corollary: $X = \binom{[n]}{2}$, $\mathcal{H} = \{\text{triangle factors } \triangle \triangle \triangle\}$,

$$p_E(\mathcal{H}) = \Theta(n^{-2/3}), \quad p_c(\mathcal{H}) = \Theta\left((\log n)^{1/3} n^{-2/3}\right).$$

(Via Riordan's coupling).

- Bounded degree spanning tree T:

Check that:

$$\left.\begin{array}{l} \cdot\ p_E(T) = \Theta(1/n), \\ \cdot\ p_c(T) \geq \Omega\left(\frac{\log n}{n}\right) \end{array}\right\} \Rightarrow p_c(T) = \Theta\left(\frac{\log n}{n}\right)$$

(Earlier proven by Montgomery 2019 via absorption)