On 3-graphs with vanishing codegree Turán density

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Abstract

For a k-uniform hypergraph (or simply k-graph) F, the codegree Turán density $\pi_{\mathrm{co}}(F)$ is the supremum over all α such that there exist arbitrarily large n-vertex F-free k-graphs H in which every $(k-1)$-subset of $V(H)$ is contained in at least αn edges. In this paper, we study the problem of what 3-graphs F satisfy $\pi_{\mathrm{co}}(F) = 0$. We find that this is closely related to the uniform Turán density $\pi_{\therefore}(F)$, which is the supremum over all d such that there are infinitely many F-free k-graphs H satisfying that any induced linear-size subhypergraph of H has edge density at least d.

We prove that, for every 3-graph F, $\pi_{\mathrm{co}}(F) = 0$ implies $\pi_{\therefore}(F) = 0$. We also introduce a layered structure for 3-graphs which allows us to obtain the reverse implication: every layered 3-graph F with $\pi_{\therefore}(F) = 0$ satisfies $\pi_{\mathrm{co}}(F) = 0$. Along the way, we answer in the negative a question of Falgas-Ravry, Pikhurko, Vaughan and Volec [J. London Math. Soc., 2023] about whether $\pi_{\therefore}(F) \leq \pi_{\mathrm{co}}(F)$ always holds. In particular, we construct counterexamples F with positive but arbitrarily small $\pi_{\mathrm{co}}(F)$ while having $\pi_{\therefore}(F) \geq 4/27$.

Our proof relies on a random geometric construction, graph distributions, Ramsey's theorem and a new formulation of the characterization of 3-graphs with vanishing uniform Turán density due to Reiher, Rödl and Schacht [J. London Math. Soc., 2018].

1 Introduction

Given a k-uniform hypergraph F (or simply *k-graph*), the *Turán number* of F, denoted by $\mathrm{ex}(n, F)$, is the maximum number of edges in an n-vertex k-graph H containing no copy of F. Within the field of extremal combinatorics, Turán-type problems represent one of the most important topics of study, dating back to the theorems of Mantel and Turán in the early 20th century. In the decades since, Turán-type problems have found applications and numerous connections in other fields, ranging from error-correcting codes in information theory to additive number theory to sphere packing, just to name a few. Targeting on the limit behavior, one may

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define the *Turán density*

$$\pi(F) := \lim_{n\to\infty} \frac{\mathrm{ex}(n,F)}{\binom{n}{k}}.$$

A standard averaging argument shows that this limit always exists. While Turán densities are well-understood for graphs (i.e., 2-graphs), determining the Turán density of a k-graph becomes notoriously difficult when $k \geq 3$. Despite much effort and countless attempts, even the Turán densities of the 3-graphs on four vertices with three and four edges, denoted by $K_4^{(3)-}$ and $K_4^{(3)}$ respectively, are still unknown. Determining the value of $\pi(K_4^{(3)})$ is one of the major open problems in extremal combinatorics, with Erdős [8] offering a \$500 reward for a solution.

The Turán density $\pi(F)$ can also be viewed asymptotically as the largest (normalized) minimum degree of an F-free k-graph, i.e. the supremum over all d such that there are arbitrarily large n-vertex F-free k-graphs H in which every vertex is contained in at least $d\binom{n}{k-1}$ edges in H. A natural variation of the Turán density, introduced by Mubayi and Zhao [22], is the codegree Turán density defined as follows. Given a k-graph H, the *degree* $d_H(S)$ of a set of vertices S is the number of edges containing it. The *minimum codegree* $\delta_{\mathrm{co}}(H)$ of H is the minimum of $d_H(S)$ over all $(k-1)$-subsets S of $V(H)$. The *codegree Turán number* $\mathrm{ex}_{\mathrm{co}}(n,F)$ is the maximum $\delta_{\mathrm{co}}(H)$ an n-vertex F-free k-graph H can admit, and the *codegree Turán density* $\pi_{\mathrm{co}}(F)$ is defined as

$$\pi_{\mathrm{co}}(F) = \lim_{n\to\infty} \frac{\mathrm{ex}_{\mathrm{co}}(n,F)}{n}.$$

This limit always exists ([22]), and it is not hard to see that $\pi_{\mathrm{co}}(F) \leq \pi(F)$ for any k-graph F. In particular, $\pi_{\mathrm{co}}(F) = \pi(F)$ when F is a graph.

For $k \geq 3$, similar as the Turán density, determining the codegree Turán density of a k-graph seems also very difficult in general. In the late 1990s, Nagle [23] and Czygrinow and Nagle [5] conjectured that $\pi_{\mathrm{co}}(K_4^{(3)-}) = \frac{1}{4}$ and $\pi_{\mathrm{co}}(K_4^{(3)}) = \frac{1}{2}$, respectively. The $\pi_{\mathrm{co}}(K_4^{(3)-})$ case was only recently settled by Falgas-Ravry, Pikhurko, Vaughan and Volec [12] via the flag algebra technique. There are few sporadic 3-graphs whose codegree Turán densities are known: the Fano plane [21], $F_{3,2}$ with $V(F_{3,2}) = [5]$ and $E(F_{3,2}) = \{123, 124, 125, 345\}$ [11], and $C_\ell^{(3)-}$ with $\ell \geq 5$, the tight cycle of length ℓ with one edge removed [24].

Recently, Piga and Schülke [25] showed that surprisingly the codegree Turán density can be arbitrarily close to zero for k-graphs when $k \geq 3$. Among all known variations of Turán density [1, 15, 20, 26, 29], this is the first example with zero being an accumulation point. For instance, there is no k-graph with its Turán density and positive codegree Turán density lying in $(0, k!/k^k)$ and $(0, 1/k)$ respectively, and no 3-graph with its uniform Turán density lying in $(0, 1/27)$. So it would be very interesting if one can characterize all k-graphs with zero codegree Turán density.

Problem 1.1. *For $k \geq 3$, characterize all k-graphs F with $\pi_{\mathrm{co}}(F) = 0$.*

Based on the result of Piga and Schülke [25], Problem 1.1 is likely to be very challenging as one cannot directly mimic the known characterization of the usual Turán density or other variations being zero by a *single* lower bound construction to avoid all k-graphs with positive codegree Turán densities.

In this paper, we mainly investigate Problem 1.1 for 3-graphs and our main contribution includes the following two parts. Firstly, we give a necessary condition for a 3-graph with vanishing codegree Turán density using its uniform Turán density. Secondly, introducing an additional layered structure condition (which we conjecture is also necessary), we provide a sufficient condition for a 3-graph having a vanishing codegree Turán density.

1.1 A necessary condition

For real numbers $d \in [0,1]$ and $\eta > 0$, an n-vertex k-graph H is said to be *uniformly (d,η)-dense* if for all $U \subseteq V(H)$, it holds that $\left|\binom{U}{k} \cap E(H)\right| \geq d\binom{|U|}{k} - \eta n^k$. Given a k-graph F, the *uniform*

Turán density of F is defined as

$$\pi_{\therefore}(F) = \sup\{d \in [0,1] : \text{for every } \eta > 0 \text{ and } n_0 \in \mathbb{N}, \text{ there exists an } F\text{-free,} \\ \text{uniformly } (d,\eta)\text{-dense } k\text{-graph } H \text{ with } |V(H)| \geq n_0\}.$$

Erdős and Sós [9] first considered the uniform Turán problems for 3-graphs and conjectured that $\pi_{\therefore}(K_4^{(3)-}) = \frac{1}{4}$. This conjecture was recently confirmed by Glebov, Král' and Volec [14] using flag algebra, and later by Reiher, Rödl and Schacht [31] via the hypergraph regularity method. For $K_4^{(3)}$, a construction due to Rödl [32] shows that $\pi_{\therefore}(K_4^{(3)}) \geq \frac{1}{2}$, but whether $\frac{1}{2}$ is the correct value of $\pi_{\therefore}(K_4^{(3)})$ still remains open. Since then, the hypergraph regularity method has been widely used in this area and many results have been obtained [3, 4, 13, 18, 29]. In particular, Reiher, Rödl and Schacht [29] characterized all 3-graphs with vanishing uniform Turán density, and provided a construction showing that the uniform Turán density cannot lie in the interval $(0, \frac{1}{27})$. For more results and problems on this topic and other variants, we refer the readers to [2, 6, 7, 17, 19, 27, 28, 30].

Our first result provides a necessary condition for a 3-graph having vanishing codegree Turán density using its uniform Turán density, and implies the relationships in Figure 1.

Theorem 1.2. *Let F be a 3-graph. If $\pi_{\mathrm{co}}(F) = 0$, then $\pi_{\therefore}(F) = 0$.*

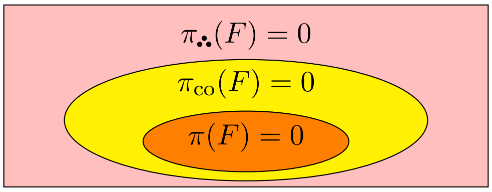

Figure 1: The relationships among Turán density, codegree Turán density and uniform Turán density for 3-graphs.

The following closely related question was recently raised by Falgas-Ravry, Pikhurko, Vaughan and Volec [12].

Question 1.3 ([12])**.** *For any 3-graph F, is it true that $\pi_{\therefore}(F) \leq \pi_{\mathrm{co}}(F)$?*

Falgas-Ravry and Lo [10] provided a positive answer to this question under a stronger uniform denseness assumption. We answer Question 1.3 in the negative by providing an infinite sequence of counterexamples.

Theorem 1.4. *For every $\varepsilon > 0$, there exists a 3-graph F with $\pi_{\therefore}(F) \geq 4/27$ and $\pi_{\mathrm{co}}(F) \leq \varepsilon$.*

Note that Theorem 1.2 together with Theorem 1.4 provides an alternative proof of the result of Piga and Schülke [25] that the codegree Turán density can be arbitrarily close to zero for 3-graphs.

1.2 A sufficient condition

At the moment, there are very few known non-trivial (i.e. not tripartite) examples of 3-graphs with zero codegree Turán density. These are limited to tight cycles of length $\ell \geq 5$ with one edge removed, $C_\ell^{(3)-}$, proved by by Piga, Sales, and Schülke [24], and zycles of length $\ell \geq 3$ with one edge removed, $Z_\ell^{(3)-}$, proved by Piga and Schülke [25]. Here, the *zycle* of length ℓ is defined as the 3-graph F with $V(F) = \cup_{i=1}^{\ell}\{v_i, u_i\}$ and $E(F) = \left(\cup_{i=1}^{\ell-1}\{u_iv_iu_{i+1}, u_iv_iv_{i+1}\}\right) \cup \{u_\ell v_\ell u_1, u_\ell v_\ell v_1\}$.

The second part of our work proves that $\pi_{\mathrm{co}}(F) = 0$ is equivalent to $\pi_{\therefore}(F) = 0$ for a large class of 3-graphs F, which we call layered 3-graphs and are defined as follows with a hierarchical structure and some kind of "1-degenerateness". In particular, our result generalizes those in [24, 25]: the above two known examples of 3-graphs with zero codegree Turán density are layered 3-graphs.

A 3-graph F is called *layered* if there exists a function $f : V(F) \to \mathbb{N}$ such that the following conditions hold.

(**A1**) In each edge uvw there is one vertex whose label is strictly greater than the other two.

(**A2**) If two edges uvw and $u'v'w'$ satisfy $\max\{f(u), f(v), f(w)\} = \max\{f(u'), f(v'), f(w')\}$, then the labels in both edges are a permutation of each other.

(**A3**) If two edges uvw and $u'v'w'$ satisfy $f(u) = f(u')$ and $f(v) = f(v')$, then $f(w) = f(w')$.

We call f a *layered function* of F, and call the set of vertices $v \in V(F)$ such that $f(v)$ is the i-th smallest integer in the range of f the *i-th layer* of F.

The second main result of our work is as follows.

Theorem 1.5. *If F is a layered 3-graph, then $\pi_{\text{co}}(F) = 0$ if and only if $\pi_{\therefore}(F) = 0$.*

Theorem 1.5 gives a sufficient condition for a 3-graph to have vanishing codegree Turán density: if a 3-graph F is layered and satisfies $\pi_{\therefore}(F) = 0$, then $\pi_{\text{co}}(F) = 0$. While $\pi_{\therefore}(F) = 0$ is indispensable by Theorem 1.2, we believe that the layered structure is also necessary. We propose the following conjecture characterizing 3-graphs with vanishing codegree Turán density.

Conjecture 1.6. *For a 3-graph F, $\pi_{\text{co}}(F) = 0$ if and only if F is layered and satisfies $\pi_{\therefore}(F) = 0$.*

It is known that $\pi_{\therefore}(F) = 0$ holds for any linear 3-graph F (see Theorem 3.1). So a special case of Conjecture 1.6 is the following.

Conjecture 1.7. *For a linear 3-graph F, $\pi_{\text{co}}(F) = 0$ if and only if F is layered.*

We show that this seemingly weaker conjecture is in fact equivalent to Conjecture 1.6, which suggests that the linear 3-graph case might be crucial for resolving Conjecture 1.6.

Theorem 1.8. *Conjecture 1.6 and Conjecture 1.7 are equivalent.*

We now present some applications of Theorem 1.5. The first one characterizes 3-graphs in a special family with vanishing codegree Turán density. A layered 3-graph F on two layers is said to be a $(2,1)$*-type 3-graph*, that is, $V(F)$ can be partitioned into two parts such that each edge of F has two vertices in one part (i.e. the first layer) and one vertex in the other part (i.e. the second layer). As the simplest layered 3-graphs, $(2,1)$-type 3-graphs play a pivotal role in both of our main results Theorem 1.2 and Theorem 1.5. We need some definitions. Given a vertex u in a 3-graph F, its *link graph*, denoted by $L_F(u)$, is the graph with $V(L_F(u)) = V(F) \setminus \{u\}$ and $E(L_F(u)) = \{vw : uvw \in E(F)\}$. Let S be a finite set, we say that σ is a *labeling* of S if $\sigma : S \to [|S|]$ is a bijection. Let G be a graph and σ be a labeling of $V(G)$. Let $u, v, w \in V(G)$ and uvw form a path of length two in G. We say that uvw is a *monotone* P_3 if $\sigma(u) < \sigma(v) < \sigma(w)$ or $\sigma(u) > \sigma(v) > \sigma(w)$.

It is well known that for a 3-graph F, $\pi(F) = 0$ if and only if F is tripartite. Note that a tripartite 3-graph is of $(2,1)$-type. Together with Lemma 3.6, Theorem 1.5 implies the following characterization of $(2,1)$-type 3-graphs F with $\pi_{\text{co}}(F) = 0$.

Corollary 1.9. *For a $(2,1)$-type 3-graph F, $\pi_{\text{co}}(F) = 0$ if and only if there is a labeling of the first layer of F such that $L_F(v)$ contains no monotone P_3 for every vertex v in the second layer.*

Note that if a graph has no monotone P_3 under a labeling, then the graph must be bipartite. So for a $(2,1)$-type 3-graph F with $\pi_{\text{co}}(F) = 0$, the link graphs $L_F(v)$ are bipartite for all vertices v in the second layer (the bipartitions could be different). Thus, Corollary 1.9 implies that every linear $(2,1)$-type 3-graph F, e.g. Fano plane with one edge removed, satisfies $\pi_{\text{co}}(F) = 0$.

In the next application, we use Theorem 1.5 and Corollary 1.9 to recover the results for $C_\ell^{(3)-}$ and $Z_r^{(3)-}$ in [24, 25].

Corollary 1.10. *For $\ell \geq 5$ and $r \geq 3$, $\pi_{\text{co}}(C_\ell^{(3)-}) = \pi_{\text{co}}(Z_r^{(3)-}) = 0$.*

Proof. As indicated in [24], every $C_\ell^{(3-)}$ with $\ell \geq 5$ is contained in a blow-up of $C_5^{(3)-}$, so we only need to show that $\pi_{\text{co}}(C_5^{(3)-}) = 0$ as any hypergraph and its blow-up have the same codegree Turán density (see e.g. [22]). Now one can easily check that $C_5^{(3)-}$ is a $(2,1)$-type 3-graph satisfying the condition in Corollary 1.9.

For $Z_r^{(3)-}$ with $r \geq 3$, suppose

$$V\left(Z_r^{(3)-}\right) = \cup_{i=1}^r \{v_i, u_i\} \text{ and } E\left(Z_r^{(3)-}\right) = \left(\cup_{i=1}^{r-1}\{u_i v_i u_{i+1}, u_i v_i v_{i+1}\}\right) \cup \{u_r v_r u_1\}.$$

Let $\sigma(u_i) = 2i - 1$ and $\sigma(v_i) = 2i$ for $1 \leq i \leq r$, and define

$$f(u) = \begin{cases} r, & \text{if } u = u_1; \\ r+1, & \text{if } u = v_1; \\ i-1, & \text{if } u \in \{u_i, v_i\} \text{ and } 2 \leq i \leq r. \end{cases}$$

It is not hard to verify that σ is a labeling satisfying (**B2**) in Theorem 3.1 and f is a layered function. Therefore, $\pi_{\text{co}}(Z_r^{(3)-}) = 0$ follows from Theorem 1.5. □

Our approach. To prove Theorem 1.2, a natural approach is to utilize a vanishing uniform Turán condition (**B2**) by Reiher, Rödl and Schacht [29] (see Theorem 3.1), and look for hypergraphs satisfying (**B2**) and linear minimum codegree. This, however, fails to work as there are uniformly dense hypergraphs not containing any subhypergraph with linear minimum codegree, see the discussion in Section 3.1. Instead of working with (**B2**), we observe a reformulation of (**B2**) using monotone paths in the link graphs (Lemma 3.4) and a variant for $(2,1)$-type 3-graphs (Lemma 3.6). A key step in our proof is to show that we may further assume that the forbidden $(2,1)$-type 3-graph F possesses certain connectedness condition (Lemma 3.9), which allows us to force a clustering phenomenon (Claim 3.11) in our random geometric $(2,1)$-type construction to avoid F. We then link this $(2,1)$-type 3-graph cyclically to obtain the final construction.

To prove Theorem 1.4, we utilize the so-called tensor product and the fact that the product is contained in large blow-ups of any component. We then observe that for any 3-graph F with minimum codegree at least two, there is a supersaturation phenomenon for F in 3-graphs with linear minimum codegree. On the other hand, such F has positive uniform Turán density. We can then take tensor product of such F to obtain counterexamples.

To prove Theorem 1.5, we first note a natural connection between certain half-bipartite graphs and the vanishing condition in Lemma 3.4, as well as a connection between graph distributions and linear codegree condition. Then applications of Ramsey's theorem show that in dense graph distributions any half-bipartite graph on a fixed vertex set will appear with a positive probability (Lemmas 4.3 and 4.4). This enables us to embed any layered 3-graph on two layers and with vanishing uniform Turán density into a $(2,1)$-type 3-graph, assuming that any pair in one part of the $(2,1)$-type 3-graph has a positive degree in the other part (Lemma 4.8), which is one of the key steps in our proof. Since the $(2,1)$-type vanishing condition (Lemma 3.6) is well compatible with the layered structure (Lemmas 4.5 and 4.6), gluing in a correct way several copies of the 3-graph obtained by removing from F all vertices on the highest layer, we can inductively embed layer by layer any layered 3-graph F with $\pi_{\therefore}(F) = 0$ into any 3-graph with a positive codegree condition using Lemma 4.7 and Lemma 4.8.

Notations. Let F be a 3-graph. The *shadow graph* of F, denoted by ∂F, is the graph with $V(\partial F) = V(F)$ and $E(\partial F) = \{uw : uvw \in E(F)\}$. Let u, v be two vertices of the 3-graph F. We say that $w \in V(F)$ is a *coneighbor* of u, v if $uvw \in E(F)$. The *coneighbor set* of u, v is defined as $N_F(uv) = \{w : uvw \in E(F)\}$. For a (hyper)graph G and a set of vertices $W \subseteq V(G)$, denote by $G - W$ the (hyper)graph obtained from deleting vertices in W from G.

The following operation provides us a natural way to merge several labelings into a larger one. Let S_1, S_2 be two disjoint finite sets and let σ_1, σ_2 be two labelings of S_1, S_2 respectively. The sum of σ_1 and σ_2, denoted by $\sigma_1 \oplus \sigma_2$, is a labeling of $S_1 \cup S_2$ where

$$\sigma_1 \oplus \sigma_2(s) = \begin{cases} \sigma_1(s), & s \in S_1; \\ \sigma_2(s) + |S_1|, & s \in S_2. \end{cases}$$

For more than two labelings $\sigma_1, \sigma_2, \dots, \sigma_k$, the sum of them, denoted by $\sum_{i=1}^{k} \sigma_i$, is inductively defined by

$$\sum_{i=1}^{k} \sigma_i = \left(\sum_{i=1}^{k-1} \sigma_i \right) \oplus \sigma_k.$$

Let S be a finite set and $\sigma' : S \to \mathbb{Z}$ be an injection. We say that a labeling σ of S is *induced by* σ' if for every $s_1, s_2 \in S$, $\sigma(s_1) > \sigma(s_2)$ if and only if $\sigma'(s_1) > \sigma'(s_2)$. Obviously, σ exists and is unique.

2 A counterexample to Question 1.3

In this section, we prove Theorem 1.4 to show the existence of 3-graphs F with $\pi_{\therefore}(F) > \pi_{\mathrm{co}}(F)$. We start by constructing the 3-graph F, then show that $\pi_{\mathrm{co}}(F) \le \varepsilon$, and finally that $\pi_{\therefore}(F) \ge 4/27$.

Proof of Theorem 1.4. For each integer $k \ge 4$, let $\mathcal{F}_k$ be the family of all k-vertex 3-graphs with minimum codegree at least two. Denote the elements of $\mathcal{F}_k$ as $\{F_1, F_2, \dots, F_\ell\}$. We define the 3-graph $\widetilde{F}_k$ with vertex set $V(F_1) \times V(F_2) \times \cdots \times V(F_\ell)$, and a triple of vertices $u, v, w \in V(F)$ form an edge if for every $1 \le i \le \ell$, the i-th coordinates of the three vertices form an edge of F_i. This is sometimes referred to as the *tensor product* of the elements of $\mathcal{F}_k$. A simple but useful fact about $\widetilde{F}_k$ we shall use is that $\widetilde{F}_k$ is contained as a subgraph in a $k^{\ell-1}$-blowup of any $F_i \in \mathcal{F}_k$ where each vertex is replaced by $k^{\ell-1}$ copies.

Claim 2.1. *For every $\varepsilon > 0$, there exists k such that $\pi_{\mathrm{co}}(\widetilde{F}_k) \le \varepsilon$.*

Proof of claim. We choose parameters satisfying $1/n \ll 1/k \ll \varepsilon$. Let H be an n-vertex 3-graph with $\delta_{\mathrm{co}}(H) \ge \varepsilon n$. Let S be a set of k vertices from H, sampled uniformly at random. Given a pair of vertices $u, v \in S$, the probability that their codegree is at most 1 can be upper-bounded by $(k-2)(1-\varepsilon)^{k-3}$, since at least $k-3$ of the remaining vertices of S must be excluded from $N_H(uv)$, and there are $k-2$ ways of selecting the potential neighbor. Therefore, the probability that $\delta_{\mathrm{co}}(H[S]) \le 1$ is at most $\binom{k}{2}(k-2)(1-\varepsilon)^{k-3}$, which is less than $1/2$ for k large enough.

Therefore, there are $\binom{n}{k}/2$ ways of selecting S so that $H[S]$ is an element of $\mathcal{F}_k$. So by the pigeonhole principle there exists i such that there are at least $\binom{n}{k}/(2|\mathcal{F}_k|)$ choices of S such that $H[S] = F_i$. For n large enough, H contains a $k^{\ell-1}$-blowup of F_i which contains $\widetilde{F}_k$ as a subgraph, so we conclude that $\pi_{\mathrm{co}}(\widetilde{F}_k) \le \varepsilon$. ■

Claim 2.2. *For each $k \ge 4$, we have $\pi_{\therefore}(\widetilde{F}_k) \ge 4/27$.*

Proof of claim. Let G be a complete (2-)graph on the vertex set $[n]$, where the edges are randomly colored red with probability $2/3$ and blue with probability $1/3$. Then construct the 3-graph H on the same vertex set $[n]$ by placing an edge on the triples $r < s < t$ if rs and rt are red and st is blue. By a standard probabilistic argument, one can show that for any $\eta > 0$, with high probability, H is uniformly $(4/27, \eta)$-dense. We will show that H does not contain a copy of $\widetilde{F}_k$.

Suppose on the contrary that $\widetilde{F}_k \subseteq H$. Then by the construction of H, there is a labeling σ of $V(\widetilde{F}_k)$ and an edge coloring of $\partial\widetilde{F}_k$ such that for any $uvw \in E(\widetilde{F}_k)$ with $\sigma(u) < \sigma(v) < \sigma(w)$, uv and uw are red and vw is blue. We say that two vertices $u, v \in V(\widetilde{F}_k)$ are *disjoint* if for all $1 \le i \le \ell$, the i-th coordinates of u and v are distinct. Note that by the construction of $\widetilde{F}_k$, a pair of vertices are contained in at least two edges if and only if they are disjoint.

Let $v \in V(\widetilde{F}_k)$ be the vertex with minimum $\sigma(v)$ such that there exists a vertex u with $\sigma(u) < \sigma(v)$ which is disjoint from v. Therefore, there is a vertex w with $uvw \in E(\widetilde{F}_k)$ (and then v and w are also disjoint). Further, a vertex u', different from u, with $u'vw \in E(\widetilde{F}_k)$ exists. Since u, u', v, w are pairwise contained in some edge of $\widetilde{F}_k$, they are pairwise disjoint. Then, by the minimality of v we have $\sigma(u) < \sigma(v) < \sigma(u'), \sigma(w)$. But then we reach a contradiction that the pair vw is blue in the edge uvw while it is red in the edge $u'vw$. ■

Theorem 1.4 follows immediately from the above two claims. □

3 The necessity of vanishing uniform Turán density

In this section, we prove Theorem 1.2 by showing that $\pi_{\mathrm{co}}(F) > 0$ for any 3-graph F with $\pi_{\therefore}(F) > 0$. For this purpose, we first prepare some important properties possessed by 3-graphs F with $\pi_{\therefore}(F) > 0$ in the subsequent three sections, and then in Section 3.4 complete our proof using a random geometric construction.

3.1 A first attempt

To prove Theorem 1.2, a natural idea is to utilize the following characterization of 3-graphs with vanishing uniform Turán density due to Reiher, Rödl and Schacht [29].

Theorem 3.1 (Reiher, Rödl and Schacht, [29]). *For any 3-graph F, the following are equivalent.*

(**B1**) $\pi_{\therefore}(F) = 0$;

(**B2**) *there is a labeling σ of the vertex set $V(F)$ and a 3-coloring $\phi : \partial F \to$ {red,blue,green} such that every edge $uvw \in E(F)$ with $\sigma(u) < \sigma(v) < \sigma(w)$ satisfies*

$$\phi(uv) = \text{red}, \ \phi(uw) = \text{blue}, \ \phi(vw) = \text{green}.$$

Fix an arbitrary 3-graph F with $\pi_{\mathrm{co}}(F) = 0$. Suppose for some $\varepsilon > 0$ and infinitely many integers n, one can construct n-vertex 3-graphs H_n satisfying (**B2**) and $\delta_{\mathrm{co}}(H_n) \ge \varepsilon n$. Then as $\pi_{\mathrm{co}}(F) = 0$, F must be a subhypergraph of H_n for some sufficiently large n. Inheriting from H_n, F also satisfies condition (**B2**), implying that $\pi_{\therefore}(F) = 0$ by Theorem 3.1 as desired. Unfortunately, we note that this idea is not feasible by the following simple observation.

Observation 3.2. *Let H be any 3-graph satisfying condition* (**B2**) *in Theorem 3.1. Then $\delta_{\mathrm{co}}(H') \le 1$ for any $H' \subseteq H$.*

Proof. Clearly, H' also satisfies condition (**B2**). Let σ be a labeling of $V(H')$ and ϕ be a 3-coloring of $\partial H'$ satisfying condition (**B2**). If $d_{H'}(uv) = 0$ for the two vertices $u, v \in V(H')$ with $\sigma(u) = 1$ and $\sigma(v) = 2$, then we are done. Otherwise, let uvw be an edge of H'. Then condition (**B2**) implies that $\phi(vw) =$ green, and therefore $d_{H'}(vw) = 1$. □

By a construction in [29], for every $\eta > 0$, there is an infinite sequence of uniformly $(\frac{1}{27}, \eta)$-dense 3-graphs satisfying condition (**B2**), which together with the above observation provides a negative answer to the following question asked by Falgas-Ravry, Pikhurko, Vaughan and Volec [12].

Question 3.3. *Let $(H_n)_{n\in\mathbb{N}}$ be a sequence of uniformly dense 3-graphs with density $d > 0$. Must there exist a sequence of subhypergraphs $(H'_n)_{n\in\mathbb{N}}$ with $H'_n \subseteq H_n$, $|V(H'_n)| \to \infty$ and $\delta_{\mathrm{co}}(H'_n)/|V(H'_n)|$ bounded away from zero?*

3.2 A $(2,1)$-type vanishing condition

In our approach, we do not insist on satisfying (**B2**) globally. We will construct infinitely many 3-graphs satisfying (**B2**) locally. For this purpose, the following reformulation of Theorem 3.1 using the structure property of link graphs is more helpful for us.

Lemma 3.4. *For any 3-graph F, the following are equivalent.*

- $\pi_{\therefore}(F) > 0$;
- *for every labeling σ of $V(F)$, there is some vertex $v \in V(F)$ such that $L_F(v)$ contains a monotone P_3.*

Proof. For one side, suppose $\pi_{\therefore}(F) > 0$, and to the contrary that there exists a labeling σ of $V(F)$ such that for every $v \in V(F)$, $L_F(v)$ does not contain a monotone P_3. For any $uv \in E(\partial F)$ with $\sigma(u) < \sigma(v)$ and $w \in N_F(uv)$, let

$$\phi(uv) = \begin{cases} \text{red}, & \sigma(w) > \sigma(v); \\ \text{blue}, & \sigma(u) < \sigma(w) < \sigma(v); \\ \text{green}, & \sigma(w) < \sigma(u). \end{cases}$$

Now we show that ϕ is well defined. Otherwise, there are $w_1, w_2 \in N_F(uv)$ such that $\sigma(w_1) < \sigma(v) < \sigma(w_2)$ or $\sigma(w_1) < \sigma(u) < \sigma(w_2)$ holds, then either w_1vw_2 is a monotone P_3 in $L_F(u)$ or w_1uw_2 is a monotone P_3 in $L_F(v)$, which contradicts to our assumption. But then this coloring ϕ and the labeling σ implies $\pi_{\therefore}(F) = 0$ by Theorem 3.1, which is also a contradiction.

For the other side, suppose that for every labeling σ of $V(F)$, there is some vertex $v \in V(F)$ such that $L_F(v)$ contains a monotone P_3. If a coloring ϕ in Theorem 3.1 exists for some labeling σ of $V(F)$, then by our assumption, there is a monotone $P_3 = w_1vw_2$ with $\sigma(w_1) < \sigma(v) < \sigma(w_2)$ in $L_F(u)$ for some $u \in V(F)$. Now we consider the color of uv. If $\sigma(u) < \sigma(v)$, then $uvw_2 \in E(F)$ implies that $\phi(uv) = \text{red}$ while $uw_1v \in E(F)$ implies that $\phi(uv) = \text{blue}$ or green, a contradiction. If $\sigma(u) > \sigma(v)$, then $vuw_2 \in E(F)$ implies that $\phi(uv) = \text{red}$ or blue while $w_1vu \in E(F)$ implies that $\phi(uv) = \text{green}$, again a contradiction. Therefore, $\pi_{\therefore}(F) > 0$ follows. □

Actually, one can easily check that the above proof implies the following.

Corollary 3.5. *For any 3-graph F, a labeling σ of $V(F)$ satisfies (**B2**) if and only if $L_F(v)$ contains no monotone P_3 for every $v \in V(F)$.*

The next lemma strengthens Lemma 3.4 for $(2,1)$-type 3-graphs, which will play an important role in our remaining proof. For convenience, when we later mention a $(2,1)$-type 3-graph with respect to a partition $X \cup Y$, we always mean that each edge of this 3-graph has two vertices in X and one vertex in Y. That is, the order of X, Y in the union matters.

Lemma 3.6. *Let F be a $(2,1)$-type 3-graph with respect to $V(F) = A \cup B$. Then $\pi_{\therefore}(F) > 0$ if and only if for every labeling of A, there exists $u \in B$ such that $L_F(u)$ contains a monotone P_3.*

Proof. For one side, suppose $\pi_{\therefore}(F) > 0$. Let σ_1 be a labeling of A and σ_2 be a labeling of B, $\sigma = \sigma_1 \oplus \sigma_2$. By Lemma 3.4, there is some $u \in A \cup B$ such that $L_F(u)$ contains a monotone P_3 with respect to σ. If $u \in A$, then $L_F(u)$ is a bipartite graph on two parts A and B. In this case, any P_3 in $L_F(u)$ has two end points lying in one part and the mid point lying in the other part. Since any vertex in B has a larger label than any vertex in A, all of P_3 in $L_F(u)$ are not monotone with respect to σ. Therefore, $u \in B$ and the monotone P_3 (with respect to σ) lies in A and is also monotone with respect to σ_1.

For the other side, let σ be a labeling of $A \cup B$. Restricting σ to A induces a labeling of A, denoted by σ_0. By the assumption, there is some $u \in B$ such that $L_F(u)$ contains a monotone P_3 with respect to the labeling σ_0. Note that this monotone P_3 is also monotone with respect to the labeling σ. Hence, $\pi_{\therefore}(F) > 0$ follows from Lemma 3.4, which finishes the proof. □

3.3 Reducing to 'connected' $(2,1)$-type 3-graphs

Let F be a 3-graph. A vertex partition $A \cup B \cup C$ of $V(F)$ is said to be *good* if $F[A \cup B], F[B \cup C], F[C \cup A]$ are all $(2,1)$-type 3-graphs, and there are no other edges in F. When proving Theorem 1.2, we will use 3-graphs with a good vertex partition. The following lemma helps us to restrict our attention to $(2,1)$-type 3-graphs. It states that in a 3-graph F with $\pi_{\therefore}(F) > 0$, if $V(F)$ admits a good vertex partition, then we can always find a $(2,1)$-type subhypergraph F' with $\pi_{\therefore}(F') > 0$.

Lemma 3.7. *Let F be a 3-graph with $\pi_{\therefore}(F) > 0$. If there exists a good vertex partition $A \cup B \cup C$ of $V(F)$, then $\pi_{\therefore}(F[A \cup B]) > 0$ or $\pi_{\therefore}(F[B \cup C]) > 0$ or $\pi_{\therefore}(F[C \cup A]) > 0$.*

Proof. Let F_1, F_2 and F_3 denote the three induced $(2,1)$-type 3-graphs $F[A \cup B], F[B \cup C]$ and $F[C \cup A]$, respectively. Suppose to the contrary that $\pi_{\therefore}(F_i) = 0$ for all $1 \le i \le 3$. Then by Lemma 3.6, there are three labelings $\sigma_A, \sigma_B, \sigma_C$ of the three parts A,B,C, respectively, such that for every $a \in A$, $b \in B$ and $c \in C$, no monotone P_3 exists in $L_{F_1}(b)$, $L_{F_2}(c)$ and $L_{F_3}(a)$. Let $\sigma = \sigma_A \oplus \sigma_B \oplus \sigma_C$. Now we prove that for every $v \in V(F)$, $L_F(v)$ does not contain a monotone P_3 with respect to σ, and hence a contradiction follows from Lemma 3.4.

For any $b \in B$, note that $L_F(b)$ is the disjoint union of $L_{F_1}(b)$ and $L_{F_2}(b)$. Since there is no monotone P_3 in $L_{F_1}(b)$ with respect to the labeling σ_A, there is no monotone P_3 in $L_{F_1}(b)$ with respect to the labeling σ. On the other hand, noting that $L_{F_2}(b)$ is a bipartite graph with partition $B \cup C$, so the two endpoints of any P_3 in $L_{F_2}(b)$ are both located in one part while the mid point is located in the other part. Since $\sigma = \sigma_A \oplus \sigma_B \oplus \sigma_C$, one can easily see that no monotone P_3 exists in $L_{F_2}(b)$. Consequently, for any $b \in B$, there is no monotone P_3 in $L_F(b)$ with respect to the labeling σ. By similar arguments, the same holds for any $x \in A \cup C$ as desired. □

We will need one more reduction using the following auxiliary graph.

Definition 3.8 (Auxiliary graph Γ)**.** Let F be a $(2,1)$-type 3-graph with respect to $A \cup B$. Define an auxiliary graph Γ_F as follows: $V(\Gamma_F) = A$, and for any two vertices $u, v \in A$, uv is an edge of Γ_F if and only if there is $w \in A$ and $x \in B$ such that $uwx, vwx \in E(F)$.

In other words, two vertices in A are joined in Γ_F if they are the endpoints of a P_3 in the link graph of some vertex in B. Assisted by this auxiliary graph, we can extract from any $(2,1)$-type 3-graph F with $\pi_{\therefore}(F) > 0$ a subhypergraph F', which admits certain 'connectedness' property and satisfies $\pi_{\therefore}(F') > 0$. This 'connectedness' property of F' is a key ingredient for our construction, allowing us to prove a clusterting phenomenon in links of our construction.

Lemma 3.9. *Let F be a $(2,1)$-type 3-graph with respect to $V(F) = A \cup B$. If $\pi_{\therefore}(F) > 0$, then there is an induced subhypergraph $F' \subseteq F$ with $\pi_{\therefore}(F') > 0$ such that $\Gamma_{F'}$ is a subgraph of some connected component of Γ_F.*

Proof. Let $A_1, A_2, \ldots, A_k \subseteq A$ be the vertex sets of all the connected components of Γ_F. For each $i \in [k]$, let

$$B_i = \{b : ba_1a_2 \in E(F), b \in B, a_1, a_2 \in A_i\},$$

and $F_i = F[A_i \cup B_i]$. Then one can easily check that Γ_{F_i} is a subgraph of the connected component of Γ_F on vertex set A_i. Now it suffices to show that there is some $i \in [k]$ such that $\pi_{\therefore}(F_i) > 0$.

We prove it by contradiction. If $\pi_{\therefore}(F_i) = 0$ for all $i \in [k]$, by Lemma 3.6, there is a labeling σ_i of A_i such that for every $b \in B_i$, there is no monotone P_3 in $L_{F_i}(b)$ with respect to σ_i. Setting $\sigma = \sum_{i=1}^{k} \sigma_i$, we next show that for every $b \in B$, there is no monotone P_3 in $L_F(b)$ with respect to σ, which contradicts to Lemma 3.6.

Let xay be a P_3 in $L_F(b)$ for some $b \in B$ with $x \in A_i, a \in A_j, y \in A_\ell$. By the definition of Γ_F, xy is an edge of Γ_F, and therefore $i = \ell$. If $i = j$, then $x, a, y \in A_i$, $b \in B_i$ and xay can not be a monotone P_3 with respect to σ since it is not monotone with respect to σ_i. If $i \neq j$, then either $\sigma(a) > \max\{\sigma(x), \sigma(y)\}$ or $\sigma(a) < \min\{\sigma(x), \sigma(y)\}$, and therefore xay is again not a monotone P_3. Combining these cases, we conclude that there is no monotone P_3 with respect to σ in $L_F(b)$. □

3.4 Proof of Theorem 1.2

We start with a construction for $(2,1)$-type 3-graphs.

Theorem 3.10. *For any positive integer k, there exists $\varepsilon = \varepsilon_k > 0$ such that for infinitely many integers n, there exists a $(2,1)$-type 3-graph H with respect to a vertex partition $V(H) = A \cup B$ such that $|A| = |B| = n$ and the following conditions are satisfied.*

(**C1**) *$d_H(a_1a_2) \geq \varepsilon n$ for any two vertices $a_1, a_2 \in A$;*

(**C2**) *$d_H(ab) \geq \varepsilon n$ for every $a \in A$ and $b \in B$;*

(**C3**) *H contains no copy of any 3-graph F with $|V(F)| \leq k$ and $\pi_{\therefore}(F) > 0$.*

Proof. Set $\varepsilon_k = \frac{1}{32k^2}$, and let n, q, r be integers such that $r = 4k$, $n = qr + 1$ and $n \gg r$. All the indices in the following construction are taken modulo n.

We first construct a random graph G on vertex set $\{a_0, a_1, \ldots, a_{n-1}\}$. For every $0 \leq i \leq n-1$ and $0 \leq j \leq r-1$, let

$$S_{ij} = \{a_{i+t} : t \in \{jq+1, jq+2, \cdots, (j+1)q\}\}.$$

Let $X_0, X_1, \ldots, X_{n-1}$ be n independent random variables, each of which takes a value from $\{0, 1, \ldots, r-1\}$ uniformly at random. Define our random graph G with $V(G) = \{a_0, a_1, \ldots, a_{n-1}\}$ such that for every $0 \leq i < j \leq n-1$, $a_ia_j \in E(G)$ if and only if $a_i \in S_{jX_j}$ and $a_j \in S_{iX_i}$. Note that a_ia_j forms an edge with probability $\frac{1}{r^2}$.

Let $G_0, G_1, \ldots, G_{n-1}$ be i.i.d. copies of G on the same vertex set $\{a_0, a_1, \ldots, a_{n-1}\}$. Define a $(2,1)$-type 3-graph H with vertex partition $A \cup B$ as follows: $A = \{a_0, a_1, \ldots, a_{n-1}\}$, $B = \{b_0, b_1, \ldots, b_{n-1}\}$, and $a_ia_jb_\ell \in E(H)$ if and only if $a_ia_j \in E(G_\ell)$. In other words, G_ℓ is the link graph of b_ℓ for any $0 \leq \ell \leq n-1$.

Finishing our proof, we next show that (**C1**) and (**C2**) are satisfied with high probability and (**C3**) is guaranteed by a geometric property of H.

Verifying (C1). Note that for any $a_i, a_j \in A$, $d_H(a_ia_j)$ is exactly the number of G_ℓ containing a_ia_j as an edge. Since $\mathbb{P}(a_ia_j \in G) = \frac{1}{r^2}$ and G_ℓ is an i.i.d. copy of G, it follows that $d_H(a_ia_j) \sim \mathrm{Bin}(n, \frac{1}{r^2})$. It follows from Chernoff's inequality that $\mathbb{P}\left(d_H(a_ia_j) \leq \frac{n}{2r^2}\right) \leq e^{-\frac{n}{8r^2}}$. Therefore, taking a union bound over $\binom{n}{2}$ such pairs, the probability that $d_H(a_ia_j) > \frac{n}{2r^2}$ for every pair of vertices a_i, a_j is at least $1 - \binom{n}{2}e^{-\frac{n}{8r^2}}$.

Verifying (C2). Let $a_i \in A$, $b_\ell \in B$. Since $d_H(a_ib_\ell) = d_{G_\ell}(a_i)$ and G_ℓ is an i.i.d. copy of G, $d_H(a_ib_\ell)$ and $d_G(a_i)$ are identically distributed. Conditioning on the choice of X_i, for each $a_j \in S_{iX_i}$, the probability that $a_ia_j \in E(G)$ is exactly $\frac{1}{r}$ and for each $a_j \notin S_{iX_i}$, the probability that $a_ia_j \in E(G)$ is zero. Since the choice of each X_j is independent, $d_G(a_i) \sim \mathrm{Bin}(q, \frac{1}{r})$. Thus, $\mathbb{E}(d_H(a_ib_\ell)) = \frac{n}{r^2}$. It follows from Chernoff's inequality that $\mathbb{P}\left(d_H(a_ib_\ell) \leq \frac{n}{2r^2}\right) \leq e^{-\frac{n}{8r^2}}$. Therefore, taking a union bound over n^2 such pairs, the probability that $d_H(a_ib_\ell) > \frac{n}{2r^2}$ for every a_i and b_ℓ is at least $1 - n^2e^{-\frac{n}{8r^2}}$.

Verifying (C3). Let F be a 3-graph with $|V(F)| \leq k$ and $\pi_{\therefore}(F) > 0$, and suppose to the contrary that H contains a copy of F. Inheriting from the structure of H, F must be a $(2,1)$-type 3-graph. By Lemma 3.9, for some $A' \subset A$ and $B' \subset B$, the induced subhypergraph $F' := F[A' \cup B']$ of F satisfies that $\pi_{\therefore}(F') > 0$ and $\Gamma_{F'}$ is a subgraph of some connected component of Γ_F. Let us imagine that the elements of A are distributed on the unit circle of the complex plane with a_ℓ being the point $e^{2\pi\ell i/n}$. The following claim shows a 'clustering' property of F'.

Claim 3.11 (Clustering). *For every $b \in B$ and $a_j \in A$, all the neighbors of a_j in $L_H(b)$ lie in an arc of length at most $\frac{2\pi}{r}$ not containing a_j. Consequently, all the vertices of A' lie on an open half circle.*

Proof of claim. The first part of the claim follows from the constructions of the random graph G and the 3-graph H. Indeed, all the neighbors of a_j in $L_H(b)$ concentrate on the random arc S_{jX_j} of length $\frac{q-1}{qr+1} \cdot 2\pi \leq \frac{2\pi}{r}$, and note that the arc S_{jX_j} does not contain a_j.

For the second part, by the definition of Γ_F and the assumption that $F \subseteq H$, each edge xy in any component of Γ_F implies that there is a path xay in $L_H(b)$ for some $a \in A$ and $b \in B$. So by the first part, the distance of x, y on the unit circle is at most $\frac{2\pi}{r}$. Since $\Gamma_{F'}$ is a subgraph of some connected component of Γ_F and $|A'| \leq |V(F)| \leq \frac{r}{4}$, we conclude that every pair of vertices in A' has a distance less than $\frac{\pi}{2}$ on the unit circle, which implies that all the elements of A' lie on an open half circle. ■

Let $\mathcal{C}$ be an open half circle containing all the vertices of A'. Along the counter-clockwise direction of $\mathcal{C}$, we obtain an enumeration $x_1, x_2, \ldots, x_{|A'|}$ of all the vertices in A'. Let σ be the labeling of A' defined by setting $\sigma(x_j) = j$ for all $1 \leq j \leq |A'|$. Recall that $\pi_{\therefore}(F') > 0$. By Lemma 3.6, for some $b \in B'$, there is a monotone path $x_\ell x_s x_t$ in $L_{F'}(b)$ with $\ell < s < t$. By the first part of Claim 3.11, x_ℓ and x_t lie in some arc $\mathcal{R}$ of length at most $2\pi/r$ not containing x_s. Since $x_s \notin \mathcal{R}$, $\mathcal{R}$ has to contain the complement of $\mathcal{C}$, which has length more than π, a contradiction. □

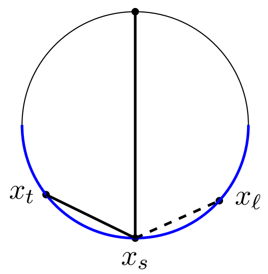

Figure 2: An illustration of the geometric property of circles that if x_t, x_s and x_ℓ lie on a (blue) half circle, then x_t and x_ℓ lie on the distinct half circles determined by x_s and its opposite point. Therefore, the construction of G implies that x_ℓ and x_t can not be joined to x_s simultaneously.

A remark here is that requiring that $n = qr + 1$ is just for our convenience to proceed the proof, our arguments actually hold for all sufficiently large integers n with a tiny modification. Combining all, we now give the proof of Theorem 1.2.

Proof of Theorem 1.2. Let F be a k-vertex 3-graph with $\pi_{\therefore}(F) > 0$. To prove Theorem 1.2, it suffices to show $\pi_{\text{co}}(F) > 0$. Let ε_k be the constant from Theorem 3.10, and let $\hat{H}$ be a $2n$-vertex $(2, 1)$-type 3-graph satisfying all (**C1**), (**C2**) and (**C3**).

Let H be a $3n$-vertex 3-graph which admits a good vertex partition $V(H) = A \cup B \cup C$ with $|A| = |B| = |C| = n$ such that $H[A \cup B], H[B \cup C]$ and $H[C \cup A]$ are all isomorphic to $\hat{H}$. By (**C1**) and (**C2**), $\delta_{\text{co}}(H) \geq \varepsilon_k n$. It remains to show that F is not a subhypergraph of H. Suppose to the contrary that H contains F. Inheriting from the structure of H, there is a good partition $A' \cup B' \cup C'$ of $V(F)$ with $A' \subseteq A, B' \subseteq B$ and $C' \subseteq C$. By Lemma 3.7, without loss of generality, we may assume that $\pi_{\therefore}(F[A' \cup B']) > 0$. Since $|A' \cup B'| \leq |V(F)| = k$, by Theorem 3.10, $F[A' \cup B']$ can not be a subhypergraph of $H[A \cup B]$, a contradiction. ☐

4 Layered 3-graphs

In this section, we prove Theorem 1.5, that is, $\pi_{\text{co}}(F) = 0$ is equivalent to $\pi_{\therefore}(F) = 0$ for every layered 3-graph F. We first introduce in Section 4.1 the notions of half-bipartite graphs and dense graph distributions and prove a lemma (Lemma 4.4) which acts as a bridge connecting $\pi_{\therefore}$ and π_{co}. Then in Section 4.2, we characterize layered 3-graphs with vanishing uniform Turán density (Lemma 4.5), and present a way of gluing layered 3-graphs (Lemma 4.6) to obtain a new one with vanishing uniform Turán density, which is a key ingredient in the induction step when proving Theorem 1.5 in Section 4.3.

4.1 Half-bipartite graphs in dense graph distributions

The first notion is a family of bipartite graphs which is relevant for the characterization of 3-graphs with vanishing uniform Turán density.

Definition 4.1. A graph G on a vertex set with a labeling σ is *half-bipartite* if it does not contain a monotone P_3. Define the *complete half-bipartite* graph B_k on $2k$ vertices to be the graph on vertices $u_1, v_1, u_2, v_2, \ldots, u_k, v_k$, in increasing order of σ, and with edges being the pairs $u_i v_j$ for all $1 \leq i \leq j \leq k$.

It is not hard to see that a half-bipartite graph must be bipartite. We first observe that the complete half-bipartite graph is a *universal* graph for all half-bipartite graphs.

Proposition 4.2. *Every half-bipartite graph F on k vertices is an ordered subgraph of B_k.*

Proof. Let the vertices of F be $w_1, w_2, \ldots, w_k$ in increasing order, then for each $i \in [k]$, map w_i to u_i if there exists $j > i$ such that $w_i w_j \in E(F)$, and map it to v_i otherwise. ☐

Note that by Lemma 3.4 a characterization of 3-graphs F with $\pi_{\therefore}(F) = 0$ is that there exists a labeling of $V(F)$ in which the link graph of every vertex is half-bipartite.

Another tool that we will use are graph distributions, which can be thought of as probability distributions on the space of graphs on a fixed set of vertices. In other words, for each graph G on a (finite) vertex set S, we assign it a non-negative real number X_G, which add up to 1. So we get a graph distribution on the vertex set S, denoted by X, by setting $\Pr(X = G) = X_G$. Further, if for some $\varepsilon > 0$, $\Pr(uv \in E(X)) \geq \varepsilon$ holds for every pair of vertices uv, then we say that X is *ε-dense*.

Let H be an n-vertex 3-graph satisfying $\delta_{\text{co}}(H) \geq \varepsilon n$. There is a simple way of obtaining an ε-dense graph distribution X out of H by selecting a vertex $v \in V(H)$ uniformly at random, and then taking $X = L_H(v)$.

So far, we have established a link between $\pi_{\therefore}$ and half-bipartite graphs, and between δ_{co} (and thus π_{co}) and ε-dense graph distributions. The following lemma connects them all.

Lemma 4.3. *For every positive integer k and real number $\varepsilon > 0$, there exist a positive integer n_0 and a real number $\delta > 0$ for which the following hold. For every ε-dense graph distribution X on vertex set $[n]$ with $n \geq n_0$, there exists a set S of $2k$ vertices such that, with probability at least δ, X contains the complete half-bipartite graph B_k in the natural order of numbers on S as a subgraph.*

Proof. We will prove this statement by induction on k. For $k = 1$ the statement is trivial, since B_1 is a single edge. Let $B'_k = B_k - u_k$. We prove the induction step in two parts. First we show that if the statement holds for B_{k-1} then it holds for B'_k, then we show that if it holds for B'_k then it holds for B_k as well.

Fix $\varepsilon > 0$. Suppose that the statement holds for $k-1$, and let δ' and n'_0 be the corresponding values of δ and n_0. Let X be an ε-dense graph distribution on $[n_1]$, where we will choose the value of n_1 later. We construct an auxiliary complete $(2k-2)$-graph H_1 on $[n_1]$ as follows. For every set S of $2k-2$ vertices, if the complete half-bipartite graph B_{k-1} on S appears in X with probability at least δ', then color the edge S of H_1 in red, otherwise color S in blue.

By the induction hypothesis, the coloring of H_1 cannot contain a blue clique of size n'_0. Therefore, by Ramsey's theorem, there is an integer n'_1 such that taking $n_1 \geq n'_1$ concludes that H_1 contains a red clique of size $2k-3+t$ for $t = \lfloor 1/\delta' \rfloor + 1$. Denote the vertices in this red clique as $u_1, v_1, u_2, v_2, \ldots, u_{k-1}, w_1, w_2, \ldots, w_t$, in increasing order. For every $1 \leq i \leq t$, the probability that the graph B_{k-1} on vertices $u_1, v_1, u_2, v_2, \ldots, u_{k-1}, w_i$ appears in X is at least δ'. These probabilities add up to $\delta' t > 1$, meaning that there exist $1 \leq i < j \leq t$ such that with probability at least $(\delta' t - 1)/\binom{t}{2} =: \delta_1$ the graph B_{k-1} appears simultaneously on $u_1, v_1, u_2, v_2, \ldots, u_{k-1}, w_i$ and $u_1, v_1, u_2, v_2, \ldots, u_{k-1}, w_j$. Then on vertices $u_1, v_1, u_2, v_2, \ldots, u_{k-1}, w_i, w_j$, a copy of B'_k appears in X with probability at least δ_1. This completes the first part of the induction step.

The second part of the induction step is similar to the first one. Take n_2 large enough, and let X be an ε-dense graph distribution on $[n_2]$. We construct a 2-coloring of the complete $(2k-1)$-graph H_2 on $[n_2]$ by coloring the edge S in red if the graph B'_k on S appears in X with probability at least δ_1, and blue otherwise. Since there cannot be a blue clique of size n'_1, by Ramsey's theorem there exists a red clique of size $2k-3+2s$ for $s = \lfloor 1/\delta_1 \rfloor + 1$. Denote the vertices of this clique as $u_1, v_1, u_2, v_2, \ldots, u_{k-2}, v_{k-2}, w_1, z_1, w_2, z_2, \ldots, w_s, z_s, v_k$. There exist $1 \leq i < j \leq s$ such that, with probability at least $\delta := (\delta_1 s - 1)/\binom{s}{2}$, the graph B'_k appears simultaneously on $u_1, v_1, u_2, v_2, \ldots, u_{k-2}, v_{k-2}, w_i, z_i, v_k$ and $u_1, v_1, u_2, v_2, \ldots, u_{k-2}, v_{k-2}, w_j, z_j, v_k$. Then a copy of B_k appears on the vertices $u_1, v_1, u_2, v_2, \ldots, u_{k-2}, v_{k-2}, w_i, z_i, w_j, v_k$ with probability at least δ. This completes the second part of the induction step. $\square$

With an extra application of Ramsey's theorem, we can find a set S of k vertices in any large ε-dense graph distribution such that every half-bipartite graph on S appears with a positive probability.

Lemma 4.4. *For every $\varepsilon > 0$ and positive integer k, there exist a positive integer n_0 and a real number $\delta > 0$ with the following property. For every ε-dense graph distribution X on $[n]$ with $n \geq n_0$, there exists a set S of k vertices on which every half-bipartite graph appears in X in the natural order of numbers with probability at least δ.*

Proof. Let $\delta = \delta_{4.3}(k, \varepsilon), n'_0 = n_{04.3}(k, \varepsilon)$ be as in Lemma 4.3. Suppose the contrary that for every set S of k vertices, there exists a half-bipartite graph F that appears with probability less than δ. On an auxiliary complete k-graph on $[n]$, assign the label F to the edge S.

Since there is a bounded number of half-bipartite graphs on k vertices, by Ramsey's theorem, if n is large enough there exists a clique K of size n'_0 where all edges receive the same label F. By the choice of n'_0, there exists a set of $2k$ vertices within this clique for which the complete half-bipartite graph B_k on these vertices appears in X with probability at least δ. But recall Proposition 4.2 that every half-bipartite graph on k vertices is a subgraph of B_k. Thus there are k vertices in K for which F appears with probability at least δ, contradicting the fact that this set of k vertices received the label F. $\square$

4.2 Layered 3-graphs with vanishing uniform Turán density

Recall that Theorem 3.1 gives a characterization of 3-graphs with vanishing uniform Turán density. But, in order to prove Theorem 1.5, it turns out to be more useful to get a specific characterization for layered 3-graphs. Let F be a layered 3-graph with k layers. For all $1 \le i < j \le k$, we denote by $F_{i,j}$ the induced subhypergraph of F on the union of the i-th layer and the j-th layer. Further, we say that $\{i, j\}$ is a *linked pair* if there exists an edge in $F_{i,j}$. Observe that by (**A1**), if $\{i, j\}$ is a linked pair, then $F_{i,j}$ is a $(2, 1)$-type 3-graph such that each edge has two vertices on the i-th layer and one vertex on the j-th layer.

Lemma 4.5. *A layered* 3*-graph F has $\pi_{\therefore}(F) = 0$ if and only if $\pi_{\therefore}(F_{i,j}) = 0$ for all linked pairs $\{i, j\}$.*

Proof. Let F be a layered 3-graph with k layers. Since $F_{i,j}$ is a subhypergraph of F, it clearly holds that if $\pi_{\therefore}(F) = 0$ then $\pi_{\therefore}(F_{i,j}) = 0$ for any linked pair $\{i, j\}$. Next we assume that $\pi_{\therefore}(F_{i,j}) = 0$ for any linked pair $\{i, j\}$ and prove that $\pi_{\therefore}(F) = 0$.

For any linked pair $\{i, j\}$ with $i < j$, since $F_{i,j}$ is a $(2, 1)$-type 3-graph, by Lemma 3.6 there exists a labeling σ_i of the i-th layer such that no vertex v on the j-th layer has a monotone P_3 in its link graph $L_{F_{i,j}}(v)$. By (**A3**), there cannot be another integer $\ell > i$ such that $\{i, \ell\}$ is also a linked pair. Therefore, the property above gives exactly one labeling of the i-th layer. For each of the remaining layers, we fix an arbitrary labeling and denote by σ_j the labeling of the j-th layer. Now we define a labeling σ of $V(F)$ by letting $\sigma = \sum_{i=1}^{k} \sigma_i$, and show that $L_F(v)$ does not contain a monotone P_3 for any $v \in V(F)$.

Let v be a vertex on the i-th layer. Then each component of $L_F(v)$ is contained in at most two layers as otherwise we can find two edges which violate (**A3**). If a component crosses two layers, say the j-th layer and the ℓ-th layer with $j < \ell$, then the component must be bipartite, because by (**A3**) each edge containing v and a vertex of the j-th layer must have the other vertex in the ℓ-th layer. Moreover, by the definition of σ, the label of each vertex in the j-th layer is smaller than the one of each vertex in the k-th layer, so there is no monotone P_3 in such component.

If a (nontrivial) component is contained in one layer, say the j-th layer, then we must have that $\{i, j\}$ is a linked pair with $j < i$. Then, because this component of $L_F(v)$ is also a component of $L_{F_{j,i}}(v)$, and $L_{F_{j,i}}(v)$ contains no monotone P_3 in the labeling σ_j, we conclude that the labeling σ produces no monotone P_3 in this component. Therefore, the link graph of each vertex does not contain a monotone P_3, and $\pi_{\therefore}(F) = 0$ follows from Lemma 3.6. $\square$

The purpose of the subsequent definitions and lemmas is to understand the circumstances under which the union of layered graphs with vanishing uniform Turán density preserves the layered structure and the uniform Turán density. This is a crucial piece in the induction step that allows us to prove Theorem 1.5 .

Let F be a layered 3-graph on k layers under a layered function f. The *reduced graph* of F, denoted by F/f, is the graph on vertex set $[k]$, in which we add a 3-edge ijk if F contains an edge whose vertices are respectively located on the i-th, j-th and k-th layers, and a directed 2-edge $\vec{ij}$ if F has an edge with two vertices on the i-th layer and one on the j-th layer. In certain sense, F/f can be seen as the result of contracting each layer of F into a single vertex.

One can easily check that if F_1 and F_2 are two layered 3-graphs on the same vertex set which share a common layered function f such that $F_1/f = F_2/f$, then f is also a layered function of the union $F_1 \cup F_2$ and $(F_1 \cup F_2)/f = F_1/f$, simply because the triples of layers containing edges of $F_1 \cup F_2$ have not changed. In fact, the same happens if for some layers, rather than identifying the corresponding vertices in F_1 and F_2, we take the disjoint union instead.

Let $F_1, F_2, \ldots, F_\ell$ be layered 3-graphs on the same vertex set which have k layers under a common layered function f. Given a subset $S \subseteq \{1, 2, \ldots, k\}$, we define $\left(\bigoplus_{i=1}^{\ell} F_i\right)/S$ as the S-*union* of these ℓ layered 3-graphs, in which for each $j \in S$ and each vertex v on the j-th layer, the corresponding copies of the vertex v in these ℓ layered 3-graphs are identified into a single vertex.

Lemma 4.6. *Let $F_1, F_2, \ldots, F_\ell$ be layered 3-graphs on the same vertex set and with the same reduced graph under a common layered function f. Suppose that $\pi_{\therefore}(F_i) = 0$ for all $1 \le i \le \ell$, and all of them share a common labeling σ satisfying* (**B2**). *Then for any set S of layers containing no linked pair, the 3-graph $F = \left(\bigoplus_{i=1}^{\ell} F_i\right)/S$ has $\pi_{\therefore}(F) = 0$.*

Proof. For every $1 \le i \le \ell$ and each $v \in V(F) \cap V(F_i)$, we use v_i to denote the corresponding copy of it in F_i. Clearly, for each $v \in V(F)$, assigning $f(v_i)$ to it if it comes from $V(F_i)$ produces a layered function of F. Therefore, by Lemma 4.5, we only need to check that $\pi_{\therefore}(F_{r,s}) = 0$ for any linked pair $\{r, s\}$.

The condition in the statement tells us that $\{r, s\} \not\subseteq S$. If neither r nor s is in S, then $F_{r,s}$ is just the disjoint union of all the 3-graphs $(F_i)_{r,s}$, and therefore $\pi_{\therefore}(F_{r,s}) = 0$. Next we assume that exactly one of $\{r, s\}$ is in S. If $r \in S$, then let σ_r be the labeling of the r-th layer of F which is induced by σ. Note that for every vertex v on the s-th layer of F, there exists a unique t such that $v \in V(F_t)$. Then $L_{F_{r,s}}(v) = L_{(F_t)_{r,s}}(v_t)$, which contains no monotone P_3 under σ, and therefore no monotone P_3 under σ_r. On the other hand, if $s \in S$, then for all $1 \le i \le \ell$, let $\sigma_{i,r}$ be the labeling of the r-th layer of F_i which is induced by σ, and let $\sigma_r = \sum_{i=1}^{\ell} \sigma_{i,r}$ be the labeling of the r-th layer of F. For each vertex v on the s-th layer of F, the link graph $L_{F_{r,s}}(v)$ is the disjoint union of all the link graphs $L_{(F_i)_{r,s}}(v_i)$, each of which contains no monotone P_3 under σ, and therefore no monotone P_3 under σ_r. We conclude that in all three cases $\pi_{\therefore}(F_{r,s}) = 0$. $\square$

4.3 Putting things together

We are almost ready to prove Theorem 1.5. The last piece of the puzzle comes from finding a way to build F layer by layer. In our case, this will mean that on each step we build a tripartite 3-graph on top of two existing layers, or a $(2, 1)$-type 3-graph on top of an existing layer. We will use the two following lemmas for these purposes.

Lemma 4.7. *For every $\varepsilon > 0$ and integer t, there exists an integer m with the following property. If H is a tripartite 3-graph with vertex partition $U \cup V \cup W$, each with size at least m, and for every $u \in U$ and $v \in V$ we have $d(u, v) \ge \varepsilon|W|$, then $K^{(3)}_{t,t,t}$ is a subgraph of H.*

Proof. By double counting on pairs (u, v), we see that H has at least $\varepsilon|U||V||W|$ edges. By a simple probabilistic argument, there exist sets $U' \subseteq U$, $V' \subseteq V$, $W' \subseteq W$, each with size exactly m, spanning at least εm^3 edges. This means that the 3-graph on these vertices has density at least $2\varepsilon/9$. The lemma follows from the fact that $\pi(K^{(3)}_{t,t,t}) = 0$. $\square$

Lemma 4.8. *For every $(2, 1)$-type 3-graph F with $\pi_{\therefore}(F) = 0$ and every $\varepsilon > 0$, there exists an integer m with the following property. If H is a $(2, 1)$-type 3-graph with vertex partition $U \cup V$, each with size at least m, and for each pair $u_1, u_2 \in U$ we have $d_H(u_1, u_2) \ge \varepsilon|V|$, then F is a subgraph of H.*

Proof. Let $U' \cup V'$ be a corresponding vertex partition of $V(F)$ with $|U'| = k$. Take an ordering of the vertices of U, and consider the graph distribution X on U, obtained by randomly sampling a vertex $v \in V$ and taking $X = L_H(v)$. Since $d_H(u_1, u_2) \ge \varepsilon|V|$, we have that X is ε-dense. Therefore, by Lemma 4.4, there exists $\delta = \delta_{4.4}(k, \varepsilon)$ such that, if m is large enough, there exists a set $S \subseteq U$ of size k for which every half-bipartite graph on S appears in X with probability at least δ.

Since $\pi_{\therefore}(F) = 0$, there exists a labeling of U' in which no vertex v of V' contains a monotone P_3 in its link graph, in other words, $L_F(v)$ is a half-bipartite graph on U' for each $v \in V'$. Now we identify the vertices of U' with the vertices of S, in the same order. If $|V| \geq m \geq |V'|/\delta$, then for each $v \in V'$, there exist at least $|V'|$ vertices of V each of which has $L_F(v)$ as a subgraph of its link graph. Thus we can select the image of the vertices of V' one by one from V, ensuring that we do not select the same vertex twice, to complete the copy of F in H. □

We are now ready to prove Theorem 1.5.

Proof of Theorem 1.5. Let F be a layered 3-graph on k layers with $\pi_{\therefore}(F) = 0$. We proceed our proof by induction on k to show that $\pi_{\mathrm{co}}(F) = 0$. If $k = 1$, then by (**A1**) the 3-graph F has no edge, and so $\pi_{\mathrm{co}}(F) = 0$ holds trivially. Now we assume that $k \geq 2$, and the induction hypothesis is that all layered 3-graphs F' on $k-1$ layers with $\pi_{\therefore}(F') = 0$ satisfy $\pi_{\mathrm{co}}(F') = 0$. Given $\varepsilon > 0$, we choose

$$1/n \ll 1/m \ll \varepsilon, 1/|V(F)|.$$

Let H be an n-vertex 3-graph with $\delta_{\mathrm{co}}(H) \geq \varepsilon n$. Our goal is to show that H contains F as a subgraph.

As $\pi_{\therefore}(F) = 0$, we can fix a labeling σ of F as in Lemma 3.4 so that the link graph of every vertex does not contain a monotone P_3. For $1 \leq i \leq k$, denote by L_i the i-th layer of F. Let $\widetilde{F}$ be the layered 3-graph on $k-1$ layers obtained from F by deleting all vertices in L_k. Clearly, $\pi_{\therefore}(\widetilde{F}) = 0$, and so $\pi_{\mathrm{co}}(\widetilde{F}) = 0$ follows from our induction hypothesis. If no edge in F intersects L_k, then $\pi_{\mathrm{co}}(F) = \pi_{\mathrm{co}}(\widetilde{F}) = 0$. So we assume that at least one edge intersects L_k, and then by (**A1**), each such edge contains a unique vertex in L_k. Moreover, by (**A2**), the layers L_i and L_j containing the other two vertices in each such edge are the same. We consider the two following cases to finish our proof.

Case 1: $i = j$. Note that $\{i, k\}$ is a linked pair and $\pi_{\therefore}(F_{i,k}) = 0$. Setting $F_{i,k}$ and $\varepsilon/2$ into Lemma 4.8 produces a value of m. Suppose that $L_i = \{v_1, v_2, \ldots, v_t\}$ with $\sigma(v_1) < \sigma(v_2) < \cdots < \sigma(v_t)$. For every t-subset S of $[m]$ with $S = \{s_1, s_2, \ldots, s_t\}$ and $s_1 < s_2 < \cdots < s_t$, we construct a 3-graph $\widetilde{F}_S$ on $k-1$ layers from $\widetilde{F}$ by replacing L_i with $[m]$ such that s_ℓ is a copy of v_ℓ for every $1 \leq \ell \leq t$ and r is an isolated vertex for each $r \in [m] \setminus S$.

Consider the 3-graph $F' = \left(\bigoplus_{S \in \binom{[m]}{t}} \widetilde{F}_S\right) / \{i\}$. Observe that by Lemma 4.6, F' is a layered 3-graph on $k-1$ layers with $\pi_{\therefore}(F') = 0$. Thus $\pi_{\mathrm{co}}(F') = 0$ follows from our induction hypothesis, implying that H contains a copy of F' if n is large enough. Now let U be the vertices of H corresponding to the i-th layer of F' (which, remember, is $[m]$), and let V be the set of vertices of H not in the copy of F'. If n is large enough, then we have both $|V| \geq m$ and every pair of vertices in U has codegree in V at least $\varepsilon n/2$. By the choice of m, there exists a copy of $F_{i,k}$ between U and V. If S is the set of vertices of $[m]$ corresponding to the vertices of U in the copy of $F_{i,k}$, then the copy of F_S together with the copy of $F_{i,k}$ form a copy of F in H, as we wanted.

Case 2: $i \neq j$. Suppose that $L_i = \{u_1, u_2, \ldots, u_{t_1}\}$ with $\sigma(u_1) < \sigma(u_2) < \cdots < \sigma(u_{t_1})$ and $L_j = \{w_1, w_2, \ldots, w_{t_2}\}$ with $\sigma(w_1) < \sigma(w_2) < \cdots < \sigma(w_{t_2})$. Setting $\varepsilon/2$ and $t = \max\{|L_i|, |L_j|, |L_k|\}$ into Lemma 4.7 produces a value of m. Let $\mathcal{X} = [m]$ and $\mathcal{Y} = [2m] \setminus [m]$. For every t_1-subset $X = \{x_1, x_2, \ldots, x_{t_1}\} \subseteq \mathcal{X}$ and every t_2-subset $Y = \{y_1, y_2, \ldots, y_{t_2}\} \subseteq \mathcal{Y}$ with $x_1 < x_2 < \cdots < x_{t_1}$ and $y_1 < y_2 < \cdots < y_{t_2}$, we construct a 3-graph $\widetilde{F}_{X,Y}$ on $k-1$ layers from $\widetilde{F}$ by replacing L_i with $\mathcal{X}$ and replacing L_j with $\mathcal{Y}$ such that x_r is a copy of u_r for every $1 \leq r \leq t_1$, y_s is a copy of w_s for every $1 \leq s \leq t_2$ and ℓ is an isolated vertex for each $\ell \in [2m] \setminus (X \cup Y)$.

Consider the 3-graph $F' = \left(\bigoplus_{(X,Y) \in \binom{\mathcal{X}}{t_1} \times \binom{\mathcal{Y}}{t_2}} \widetilde{F}_{X,Y}\right) / \{i, j\}$, which is a 3-graph on $k-1$ layers. Note also that by (**A3**), $\{i, j\}$ is not a linked pair, and hence $\pi_{\therefore}(F') = 0$ by Lemma 4.6.

By our induction hypothesis, we have $\pi_{\mathrm{co}}(F') = 0$, meaning that H contains a copy of F' if n is large enough. Let U and V be the sets of vertices on the i-th and j-th layers of this copy of F', and let W be the set of vertices of H outside of this copy of F'. If n is large enough, then we have that $|W| \geq m$ and that for all pairs $uv \in U \times V$, the codegree of uv in W is at least $\varepsilon n/2$. By the choice of m, there exists a copy of $K^{(3)}_{t,t,t}$ between U, V and W, and therefore a copy of $K^{(3)}_{t_1,t_2,|L_k|}$ with $t_1, t_2, |L_k|$ vertices from U, V, W, respectively. If X and Y are the sets of vertices of $\mathcal{X}$ and $\mathcal{Y}$ corresponding to the vertices of U and V in the copy of $K^{(3)}_{t_1,t_2,|L_k|}$, then the copy of $F_{X,Y}$ together with the copy of $K^{(3)}_{t_1,t_2,|L_k|}$ form a copy of F in H, as we wanted.

We have shown that in all cases H contains F as a subgraph, and so $\pi_{\mathrm{co}}(F) = 0$ as desired. $\square$

5 Is the layered structure necessary?

In order to prove Conjecture 1.6, one would need to prove that all hypergraphs F which are not layered satisfy $\pi_{\mathrm{co}}(F) > 0$. It seems, unlike the usual Turán density π and the uniform Turán density $\pi_{\therefore}$, there are no simple explicit constructions with linear codegree that work for *all* non-layered hypergraphs. We give an example to illustrates the difficulty in trying to find such simple explicit constructions. This example comes from gluing together two layered 3-graphs. While both layered 3-graphs are very simple and have codegree Turán density zero, the resulting graph F is not layered and the simplest constructions showing $\pi_{\mathrm{co}}(F) > 0$ we can find is already relatively complex.

5.1 A specific example

Let F_1 be the hypergraph on vertices $\{a, b, c, d, e\}$, with edges $\{abc, abd, cde\}$. Let F_2 be the hypergraph on vertices $\{e, f, g, h, i, j, k, \ell\}$ with edges $\{fgh, fgi, hij, hik, jk\ell, eh\ell\}$. Let $F = F_1 \cup F_2$.

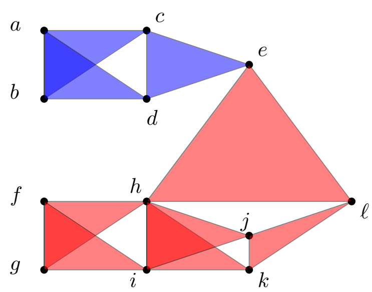

Figure 3: The hypergraph F, with F_1 in blue and F_2 in red.

Observe that F_1 and F_2 are both layered and $\pi_{\therefore}(F_1) = \pi_{\therefore}(F_2) = 0$, so by Theorem 1.5 we have $\pi_{\mathrm{co}}(F_1) = \pi_{\mathrm{co}}(F_2) = 0$. On the other hand, one can check that F is not layered.

We now give a construction showing $\pi_{\mathrm{co}}(F) \geq 1/12$, and in particular that the codegree density of F is positive. Consider the following hypergraph H with vertex set is partitioned into twelve equal parts $\mathcal{P} = \{A, B, C, D, E, F, G, H, I, J, K, L\}$, and an edge is placed if it belongs to one of the triples in

$$\begin{aligned} T = \{&AAB, \quad ACI, \quad ADG, \quad AEE, \quad AFF, \quad AHJ, \quad AKL, \quad BBC, \quad BDJ, \quad BEH, \\ &BFK, \quad BGG, \quad BIL, \quad CCD, \quad CEF, \quad CGK, \quad CHH, \quad CJL, \quad DDE, \quad DFL, \\ &DHK, \quad DII, \quad EGI, \quad EJJ, \quad EKK, \quad ELL, \quad FGJ, \quad FHI, \quad GHL, \quad IJK\}. \end{aligned}$$

A relevant property of T is that every pair of parts XY, including those with $X = Y$, is contained in exactly one triple in T. Because of this, any pair of edges uvw and uvw' in H which have two common vertices must satisfy that w and w' belong to the same part in $\mathcal{P}$. Observe also that in all triples in T of the form XXY we have $Y \in \{A, B, C, D, E\}$.

In every copy of F_1 in H, because of the pair of edges abc and abd, the vertices c and d belong to the same part of $\mathcal{P}$, which forces $e \in \{A, B, C, D, E\}$. Meanwhile, in every copy of F_2 in H, because of the pair of edges fgh and fgi we have that h and i belong to the same part of $\mathcal{P}$. This part determines the partition class of j and k through the edges hij and hik, which in turn determine the class of ℓ through $jk\ell$, which finally determines the class of e through $eh\ell$. Checking all twelve classes for h and i, we find that for none of them the resulting vertex e is in $\{A, B, C, D, E\}$, meaning that there is no copy of F in H.

We remark that vertex transitive hypergraphs with linear minimum codegree will contain F: there will be copies of F_1 and F_2, of which we can find isomorphisms in which the vertex e coincides.

5.2 Equivalence of Conjectures 1.6 and 1.7

In this section, we prove Theorem 1.8, that is, Conjectures 1.6 and 1.7 are equivalent. By Theorem 1.5, to prove the equivalence, it suffices to show that any 3-graph F with $\pi_{\mathrm{co}}(F) = 0$ is layered assuming that this holds for linear 3-graphs. For this purpose, we will use the following operation to transform a 3-graph into a linear 3-graph while preserving the (non)layered structure.

For any 3-graph F and any vertex $v \in V(F)$, let F_v be the 3-graph obtained by the following operation.

- Delete vertex v, and for each $uw \in L_F(v)$, add a new vertex v_{uw} and a new edge uwv_{uw}.
- Add three new vertices x_v, y_v, z_v, and then for each $uw \in L_F(v)$, add three more new vertices x_{uw}, y_{uw}, z_{uw} and the following six new edges

$$x_v v_{uw} x_{uw},\, x_v y_{uw} z_{uw},\, y_v v_{uw} y_{uw},\, y_v x_{uw} z_{uw},\, z_v v_{uw} z_{uw},\, z_v x_{uw} y_{uw},$$

  which is a Fano plane on $\{x_v, y_v, z_v, v_{uw}, x_{uw}, y_{uw}, z_{uw}\}$ with one edge $x_v y_v z_v$ removed and is denoted by $\mathbf{F}_{uw}$.

Denote by $\mathcal{L}_v = \{x_v, y_v, z_v\} \cup \{v_{uw}, x_{uw}, y_{uw}, z_{uw} : uw \in L_F(v)\}$ the collection of all these new vertices. Note that for each $u \in \mathcal{L}_v$, its link graph is a matching. We say that the vertex v is *linearized.*

Given a 3-graph F and a function $f : V(F) \to \mathbb{N}$, we say that f is a *semi-layered function* of F if it satisfies conditions (**A1**) and (**A2**), and define the *cardinality* of f to be the size of its range. Furthermore, a semi-layered function is *minimum* if it has the minimum cardinality over all semi-layered functions. For the sake of convenience, for any edge $e = uvw \in E(F)$, we use $f(e)$ to denote the multiset $\{f(u), f(v), f(w)\}$.

Proposition 5.1. *Every minimum semi-layered function is a layered function.*

Proof. Let F be a 3-graph and f be a minimum semi-layered function of F. Suppose on the contrary that f is not a layered function of F. Then there are two edges $e, e' \in E(F)$ satisfying that $|f(e) \cap f(e')| = 2$. Therefore, $\max f(e) \neq \max f(e')$ holds as otherwise $f(e) = f(e')$ would follow from condition (**A2**). Let $p = \max f(e)$, $t = \max f(e')$ and q be the unique element in $f(e') \setminus f(e)$. Without loss of generality, we may assume that $p > t \geq q$. For each $v \in V(F)$, let

$$g(v) = \begin{cases} f(v), & \text{if } f(v) \neq p; \\ q, & \text{if } f(v) = p. \end{cases}$$

Note that $g(e) = g(e') = f(e')$, and it is not hard to see that

($\star$) either $\max g(\hat{e}) = \max f(\hat{e})$ or $\max g(\hat{e}) = t$ for any edge $\hat{e}$ with $p \in f(\hat{e})$.

We show that g is also a semi-layered function of F, which leads to a contradiction as the cardinality of g is smaller than the one of f.

Let e_0 be an edge in $E(F)$ with $\max f(e_0) = \ell$. If $\ell > p$, then $g(e_0)$ has a unique maximum element because $p > q$ and f satisfies (**A1**). If $\ell < p$, then $g(e_0) = f(e_0)$ as $p \notin f(e_0)$, implying that $g(e_0)$ also has a unique maximum element. If $\ell = p$, then $f(e_0) = f(e)$ by condition (**A2**). Therefore, $g(e_0) = g(e) = g(e') = f(e')$, which implies that $g(e_0)$ has a unique maximum element as $f(e')$ does. So g satisfies condition (**A1**).

Let e_1, e_2 be two edges with $\max g(e_1) = \max g(e_2) = \ell$. If $\ell \neq t$, then by ($\star$) we must have $\max f(e_1) = \max f(e_2) = \ell$, implying that $f(e_1) = f(e_2)$. Therefore, $g(e_1) = g(e_2)$ follows. If $\ell = t$, then for each $i \in \{1, 2\}$, either $\max f(e_i) = t$ or $\max f(e_i) = p$, which implies that either $f(e_i) = f(e)$ or $f(e_i) = f(e')$. Therefore, $g(e_1) = g(e_2)$ follows from that $g(e_i) \in \{g(e), g(e')\}$ for each $i \in \{1, 2\}$ and $g(e) = g(e')$. So g satisfies condition (**A2**). $\square$

A straightforward corollary of Proposition 5.1 is that a 3-graph F is layered if and only if F has a semi-layered function. Now we show that linearizing a vertex preserves the (non)layered structure.

Proposition 5.2. *For any 3-graph F and $v \in V(F)$, F is layered if and only if F_v is layered.*

Proof. Suppose that F is layered, and let f be a layered function of F. Now we define a function $g : V(F_v) \to \mathbb{N}$ by setting

$$g(u) = \begin{cases} f(u), & \text{if } u \in V(F_v) \setminus \mathcal{L}_v; \\ f(v), & \text{if } u \in \mathcal{L}_v \setminus \{x_v, y_v, z_v\}; \\ N, & \text{if } u \in \{x_v, y_v, z_v\}, \end{cases}$$

where $N = \max\{f(u) : u \in V(F)\} + 1$. Next we prove that F_v is layered by showing that g is a semi-layered function of F_v.

One can easily verify that g satisfies condition (**A1**). Now we show that g also satisfies condition (**A2**). Let $e_1, e_2 \in E(F_v)$ be any two edges with $\max g(e_1) = \max g(e_2) = M$. If $M = N$, then $e_1, e_2 \subseteq \mathcal{L}_v$ and $g(e_1) = g(e_2) = \{f(v), f(v), N\}$. If $M \neq N$, then for any $i \in \{1, 2\}$, either $e_i \cap \mathcal{L}_v = \varnothing$ or $e_i = uwv_{uw}$ for some $uw \in L_F(v)$. By the definition of g, it follows that either $g(e_i) = f(e_i)$ or $g(e_i) = f(uwv)$ for some $uw \in L_F(v)$, implying that $g(e_1) = g(e_2)$ as f satisfies condition (**A2**). Therefore, g satisfies condition (**A2**).

Reversely, suppose that F_v is layered and $g : V(F_v) \to \mathbb{N}$ is a layered function of F_v. For each $uw \in L_F(v)$, since $\mathbf{F}_{uw}$ is a Fano plane with one edge removed, it is not hard to verify that the layered function g must satisfy that

$$g(x_v) = g(y_v) = g(z_v) > g(v_{uw}) = g(x_{uw}) = g(y_{uw}) = g(z_{uw}).$$

Furthermore, by condition (**A2**), we know that $g(v_1) = g(v_2)$ for any two vertices $v_1, v_2 \in \mathcal{L}_v \setminus \{x_v, y_v, z_v\}$. Let $\hat{v}$ be a vertex in $\mathcal{L}_v \setminus \{x_v, y_v, z_v\}$. Define a new function $f : V(F) \to \mathbb{N}$ by setting

$$f(u) = \begin{cases} g(\hat{v}), & \text{if } u = v; \\ g(u), & \text{if } u \in V(F) \setminus \{v\}. \end{cases}$$

Note that for any edge $uwv \in E(F)$,

$$f(uwv) = \{g(u), g(w), g(\hat{v})\} = \{g(u), g(w), g(v_{uw})\} = g(uwv_{uw}).$$

Combining with that $f(e) = g(e)$ for any $e \in E(F)$ not containing v, it clearly holds that f is a semi-layered function of F, and thus F is layered. $\square$

The following proposition shows that linearizing a vertex of any 3-graph will not increase its codegree Turán density.

Proposition 5.3. *For any 3-graph F and $v \in V(F)$, $\pi_{\rm co}(F_v) \le \pi_{\rm co}(F)$.*

Proof. Choose parameters satisfying $1/n \ll 1/t \ll \varepsilon, 1/|V(F)|$. Let $F(v,t)$ be the 3-graph obtained from F by blowing up the vertex v into an independent set of size t, then $\pi_{\rm co}(F(v,t)) = \pi_{\rm co}(F)$. Let H be any n-vertex 3-graph with $\delta_{\rm co}(H) \ge (\pi_{\rm co}(F)+\varepsilon)n$. Then $F(v,t)$ is a subgraph of H. Let U_1 be the vertex set corresponding to the blow-up of v and $U_2 = V(H) \setminus V(F(v,t))$. Let H' be the $(2,1)$-type subgraph of H with $V(H') = U_1 \cup U_2$ and $E(H') = \{e \in H : |e \cap U_1| = 2, |e \cap U_2| = 1\}$. Therefore, $d_{H'}(u_1,u_2) \ge (\pi_{\rm co}(F) + \varepsilon/2)n$ for any two vertices $u_1, u_2 \in U_1$. Further, note that $F_v[\mathcal{L}_v]$ is a $(2,1)$-type linear 3-graph with partition $\{v_{uw}, x_{uw}, y_{uw}, z_{uw} : uw \in L_F(v)\} \cup \{x_v, y_v, z_v\}$. We conclude that $F_v[\mathcal{L}_v]$ is a subgraph of H' by Lemma 4.8. Hence, F_v is a subgraph of H, implying that $\pi_{\rm co}(F_v) \le \pi_{\rm co}(F)$. □

Proof of Theorem 1.8. Note that, by Theorem 1.5, we only need to show that any 3-graph F with $\pi_{\rm co}(F) = 0$ is layered assuming that this is true for all linear 3-graphs.

Let F be a 3-graph with $\pi_{\rm co}(F) = 0$, and $\mathcal{L}(F)$ be a linear 3-graph obtained from F by linearizing, one by one, all vertices whose link graphs are not matchings. Then it follows from Proposition 5.3 that $\pi_{\rm co}(\mathcal{L}(F)) = 0$. Suppose that Conjecture 1.7 is true, then $\mathcal{L}(F)$ is layered, which implies that F is also layered by Proposition 5.2. □

6 Concluding Remarks

In this paper, we studied Problem 1.1 for 3-graphs and proved that any layered 3-graph F with $\pi_{\therefore}(F) = 0$ has $\pi_{\rm co}(F) = 0$. On the other hand, we proved that $\pi_{\therefore}(F) = 0$ is a necessary condition and reduced the problem determining whether the layered structure is necessary to the linear 3-graph case. One explanation for the difficulty of Problem 1.1 is that the codegree Turán density can be arbitrarily close to zero. Towards Conjecture 1.7, we wonder if zero is also an accumulation point for the codegree Turán density for linear 3-graphs.

Question 6.1. *For any $\varepsilon > 0$, is there a linear 3-graph F with $0 < \pi_{\rm co}(F) < \varepsilon$?*

Similar as codegree Turán density, one can define the s-degree Turán density, which was first mentioned by Keevash [16] and formally introduced by Lo and Markström [20]. Let H be an n-vertex k-graph. For $1 \le s \le k-1$, the *minimum s-degree*, denoted by $\delta_s(H)$, is the minimum of $d_H(S)$ over all s-subsets S of $V(H)$. Given a family $\mathcal{F}$ of k-graphs, the *s-degree Turán number* $\mathrm{ex}_s^k(n,\mathcal{F})$ is the largest $\delta_s(H)$ over all n-vertex k-graphs containing none of the members in $\mathcal{F}$. Similarly, the *s-degree Turán density* of $\mathcal{F}$ is defined to be

$$\pi_s^k(\mathcal{F}) = \lim_{n\to\infty} \frac{\mathrm{ex}_s^k(n,\mathcal{F})}{\binom{n}{k-s}}.$$

Lo and Markström [20] showed that this limit always exists and $\pi_s^k(F)$ also possesses the supersaturation property for any k-graph F.

Let $\widetilde{\Pi}_s^k = \{\pi_s^k(\mathcal{F}) : \mathcal{F}$ is a family of k-graphs$\}$ and $\Pi_s^k = \{\pi_s^k(F) : F$ is a k-graph$\}$. Mubayi and Zhao [22] proved that $\widetilde{\Pi}_s^k$ is dense in $[0,1)$ for $k \ge 3$ and $s = k-1$. Lo and Markström [20] later extended this result to all $2 \le s \le k-1$ (note that $\pi_1^k(\mathcal{F}) = \pi(\mathcal{F})$, and therefore $\widetilde{\Pi}_1^k$ is not dense in $[0,1)$). But the same question for Π_s^k still remains widely open.

Question 6.2. *For $k \ge 3$ and $2 \le s \le k-1$, is Π_s^k dense in $[0,1)$?*

As a positive evidence, Piga and Schülke [25] proved that zero is an accumulation point of Π_{k-1}^k for $k \ge 3$. So it would be very interesting to figure out whether zero is also an accumulation point of Π_s^k with $2 \le s \le k-2$.

Question 6.3. *For $k \ge 4$ and $2 \le s \le k-2$, is zero an accumulation point of Π_s^k?*

Generalizing Problem 1.1, one can consider the characterization problem for all $2 \leq s \leq k-1$, and it would be interesting if our results can be extended to these general cases.

Problem 6.4. *For $k \geq 3$ and $2 \leq s \leq k-1$, characterize all k-graphs F with $\pi_s^k(F) = 0$.*

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References

[1] J. Balogh, F. C. Clemen, and B. Lidický. Hypergraph Turán problems in ℓ_2-norm. *arXiv:2108.10406*, 2021.

[2] S. Berger, S. Piga, C. Reiher, V. Rödl, and M. Schacht. Turán density of cliques of order five in 3-uniform hypergraphs with quasirandom links. *arXiv:2206.07354*, 2022.

[3] M. Bucić, J. W. Cooper, D. Král', S. Mohr, and D. M. Correia. Uniform Turán density of cycles. *Transactions of the American Mathematical Society*, 376(7):4765–4809, 2023.

[4] A. Chen and B. Schülke. Beyond the broken tetrahedron. *arXiv:2211.12747*, 2022.

[5] A. Czygrinow and B. Nagle. A note on codegree problems for hypergraphs. *Bulletin of the Institute of Combinatorics and its Applications*, 32:63–69, 2001.

[6] L. Ding, J. Han, S. Sun, G. Wang, and W. Zhou. F-factors in quasi-random hypergraphs. *Journal of the London Mathematical Society (2)*, 106(3):1810–1843, 2022.

[7] L. Ding, J. Han, S. Sun, G. Wang, and W. Zhou. Tiling multipartite hypergraphs in quasi-random hypergraphs. *Journal of Combinatorial Theory, Series B*, 160:36–65, 2023.

[8] P. Erdős. On the combinatorial problems which I would most like to see solved. *Combinatorica*, 1:25–42, 1981.

[9] P. Erdős and V. T. Sós. On Ramsey–Turán type theorems for hypergraphs. *Combinatorica*, 2(3):289–295, 1982.

[10] V. Falgas-Ravry and A. Lo. Subgraphs with large minimum ℓ-degree in hypergraphs where almost all ℓ-degrees are large. *Electronic Journal of Combinatorics*, 25(2):Paper 18, 1–7, 2018.

[11] V. Falgas-Ravry, E. Marchant, O. Pikhurko, and E. R. Vaughan. The codegree threshold for 3-graphs with independent neighborhoods. *SIAM Journal on Discrete Mathematics*, 29(3):1504–1539, 2015.

[12] V. Falgas-Ravry, O. Pikhurko, E. Vaughan, and J. Volec. The codegree threshold of K_4^-. *Journal of the London Mathematical Society (2)*, 107:1660–1691, 2023.

[13] F. Garbe, D. Král', and A. Lamaison. Hypergraphs with minimum positive uniform Turán density. *Israel Journal of Mathematics*, doi:10.1007/s11856-023-2554-0, 2023.

[14] R. Glebov, D. Král', and J. Volec. A problem of Erdős and Sós on 3-graphs. *Israel Journal of Mathematics*, 211(1):349–366, 2016.

[15] A. Halfpap, N. Lemons, and C. Palmer. Positive co-degree density of hypergraphs. *arXiv:2207.05639*, 2022.

[16] P. Keevash. A hypergraph regularity method for generalized Turán problems. *Random Structures & Algorithms*, 34(1):123–164, 2009.

[17] J. Lenz and D. Mubayi. Perfect packings in quasirandom hypergraphs I. *Journal of Combinatorial Theory, Series B*, 119:155–177, 2016.

[18] H. Li, H. Lin, G. Wang, and W. Zhou. Hypergraphs with a quarter uniform Turán density. *arXiv:2305.11749*, 2023.

[19] H. Lin, G. Wang, and W. Zhou. The minimum positive uniform Turán density in uniformly dense k-uniform hypergraphs. *arXiv:2305.01305*, 2023.

[20] A. Lo and K. Markström. ℓ-degree Turán density. *SIAM Journal on Discrete Mathematics*, 28(3):1214–1225, 2014.

[21] D. Mubayi. The co-degree density of the fano plane. *Journal of Combinatorial Theory, Series B*, 95(2):333–337, 2005.

[22] D. Mubayi and Y. Zhao. Co-degree density of hypergraphs. *Journal of Combinatorial Theory, Series A*, 114(6):1118–1132, 2007.

[23] B. Nagle. Turán-related problems for hypergraphs. *Congressus Numerantium*, page 119–128, 1999.

[24] S. Piga, M. Sales, and B. Schülke. The codegree Turán density of tight cycles minus one edge. *Combinatorics, Probability and Computing*, 32:881–884, 2023.

[25] S. Piga and B. Schülke. Hypergraphs with arbitrarily small codegree Turán density. *arXiv:2307.02876*, 2023.

[26] O. Pikhurko. On the limit of the positive ℓ-degree Turán problem. *The Electronic Journal of Combinatorics*, 30(3):P3.25, 2023.

[27] C. Reiher. Extremal problems in uniformly dense hypergraphs. *European Journal of Combinatorics*, 88:103117, 2020.

[28] C. Reiher, V. Rödl, and M. Schacht. Embedding tetrahedra into quasirandom hypergraphs. *Journal of Combinatorial Theory, Series B*, 121:229–247, 2016.

[29] C. Reiher, V. Rödl, and M. Schacht. Hypergraphs with vanishing Turán density in uniformly dense hypergraphs. *Journal of the London Mathematical Society*, 97(1):77–97, 2018.

[30] C. Reiher, V. Rödl, and M. Schacht. On a generalisation of Mantel's theorem to uniformly dense hypergraphs. *International Mathematics Research Notices*, 16:4899–4941, 2018.

[31] C. Reiher, V. Rödl, and M. Schacht. On a Turán problem in weakly quasirandom 3-uniform hypergraphs. *Journal of the European Mathematical Society*, 20(5):1139–1159, 2018.

[32] V. Rödl. On universality of graphs with uniformly distributed edges. *Discrete Mathematics*, 59(1-2):125–134, 1986.