Lecture 27

- Pf Ind. Matching thm
- pts/lines incidence graphs
- Matoušek's Cutting Lem

Previously: • IMT $\Rightarrow$ (6,3)-thm

• Exer: IMT $\Rightarrow$ Roth's thm

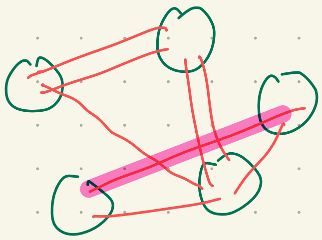
Recall

Thm (IMT) $\forall$ n-vx gr G

$E(G) = \dot{\bigcup}_{i=1}^{n} M_i$, where $\Longrightarrow$ $e(G) = o(n^2)$

M_i induced matching

Idea: • RL on G $\Rightarrow$ $G_R \subseteq G$ Cleaning $o(n^2)$ edges $\Rightarrow$ some ind. match M_i has size $\Omega(n)$ in G_R



• Consider clusters that has $\Omega(n)$ intersection with M_i

• Some edge $e \in M_i$ survives

$\Rightarrow$ e is between some large subsets of dense and reg pair

$\Rightarrow$ $\exists$ non-M_i-edge ↯.

Pf: Suppose $e(G) \geq cn^2$ for some $c > 0$.

• Take $\varepsilon = \frac{c}{10}$ & apply RL on G with $m = 1/\varepsilon$

$\Rightarrow$ $R = R(\varepsilon, 2\varepsilon)$ reduced gr

Do the cleaning $\Rightarrow$ $G_R \subseteq G$ corresponding to R

edges removed $= e(G) - e(G_R) \leq 3\varepsilon n^2$

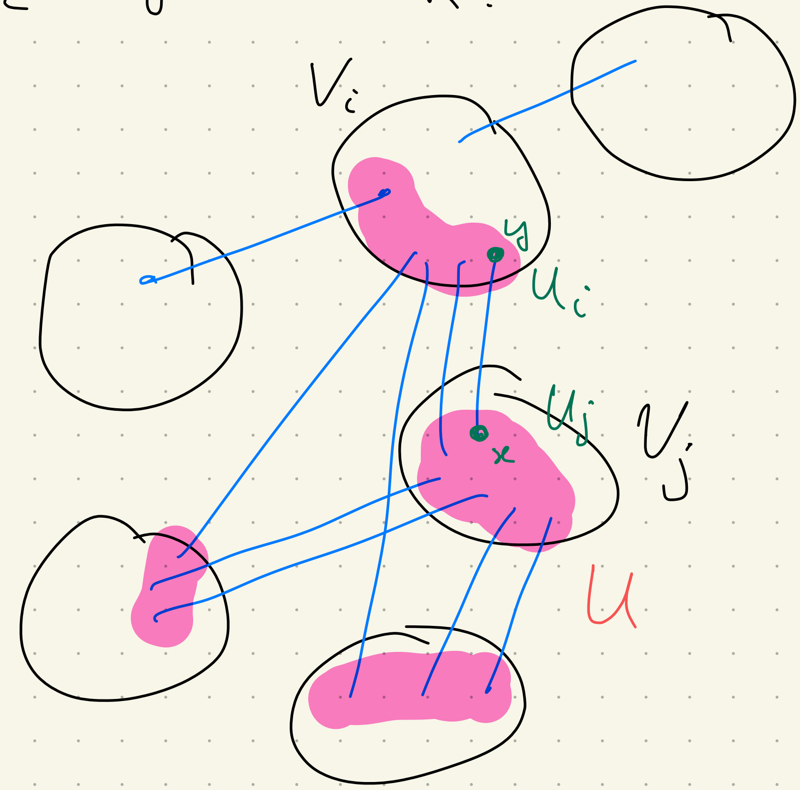
$\Rightarrow$ $e(G_R) \geq cn^2 - 3\varepsilon n^2 \geq cn^2/2$

$\Rightarrow$ by pigeonhole, $\exists$ an induced matching M

with $\geq \frac{e(G_R)}{n} \geq cn/2$ edges in G_R.

- $|V(M)| \geq 2|M| \geq cn$

Define $U_i = V_i \cap V(M)$
$\forall\, i \in [r]$



Let

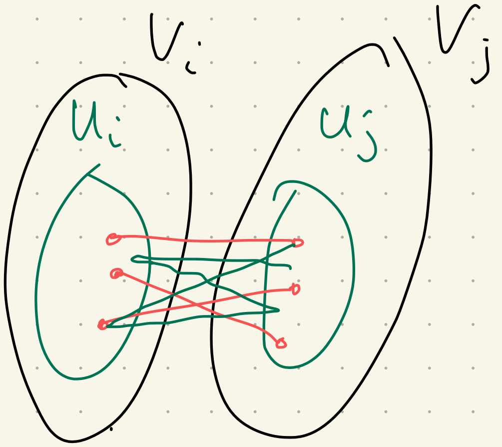
$\mathcal{U} = \bigcup \{ U_i : |U_i| \geq \varepsilon |V_i|, {}_{i \in [r]} \}$

- From $V(M) = \bigcup U_i \longrightarrow \mathcal{U}$, we discard $\leq \varepsilon n$ vxs

which is less than $\frac{cn}{2} \leq |M|$

$\Rightarrow$ some $\underset{xy}{\text{edge}}$ of M survives in $\mathcal{U}$

This edge xy lives in between
some dense & regular pair
(V_i, V_j) say.



Since $|U_i| \geq \varepsilon |V_i|$, $|U_j| \geq \varepsilon |V_j|$

$$\Rightarrow \quad d(U_i, U_j) \geq d(V_i, V_j) - \varepsilon \geq \varepsilon$$

$$\Rightarrow \quad e_G(U_i, U_j) \geq \varepsilon |U_i||U_j| = \Omega(n^2)$$

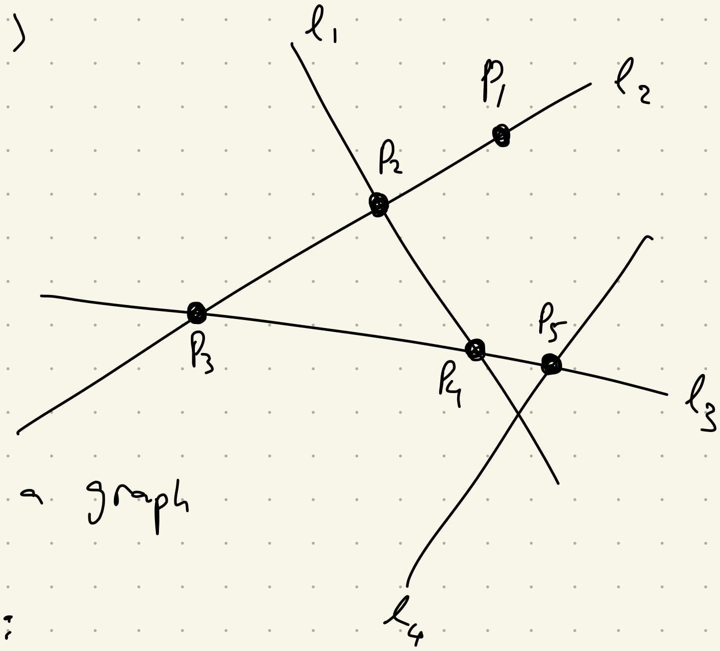
$$\gg n \geq |M|$$

$\Rightarrow$ $\exists$ a non-M-edge on $V(M) \supseteq U$ contradicting to M being an induced matching. ☺

§ C_6-free point-line incidence graph.

Given a points/lines arrangement on the plane $\mathbb{R}^2$

an incidence is a pair (p, ℓ) s.t. $p \in \ell$

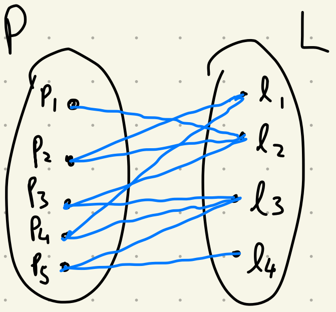


Example :

# line	# pts	# incidences
4	5	9

Record all the incidences info in a graph

Incidence graphs for (P, L) :



Bipartite graph on partite sets P & L

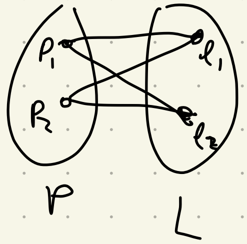
$$p \sim \ell \iff p \in \ell$$

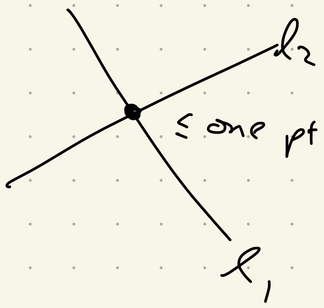
$$e(G) = \# \text{ pt/line incidences}$$

Rmk ∀ incidence graph of pt/lines on $\mathbb{R}^2$

$\Rightarrow$ C_4-free

$ex(n, C_4) = \Theta(n^{3/2})$





Thm (Szemerédi–Trotter)

∀ pts/lines arrangement with $|P| = |L| = n$

$\Rightarrow$ # incidences $= O(n^{4/3})$

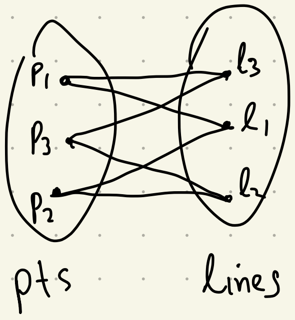
∀ incidence gr G of pts/lines $\mathbb{R}^2$ $\Rightarrow$ $e(G) = O(n^{4/3})$

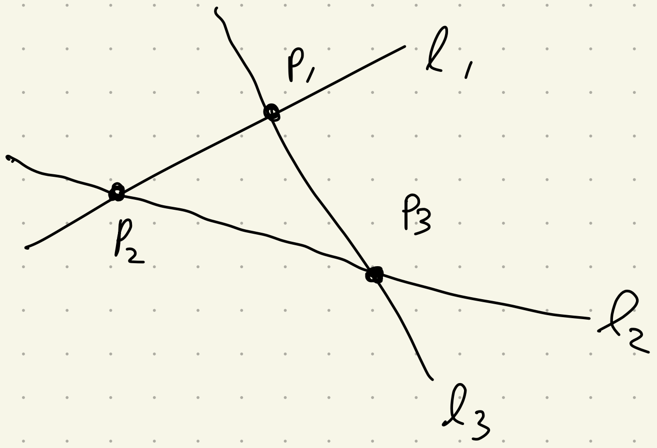
Consider now C_6-free pt/line incidence graph.

Recall $ex(n, C_6) = \Theta(n^{4/3})$

Thm ∀ (n,n)-vertex pt/line incidence gr G.

C_6-free $\Rightarrow$ $e(G) = o(n^{4/3})$





C_6 in G $\longleftrightarrow$ triangle arrangement

OPEN C_6-free $\overset{n\text{-vx}}{\text{incidence}} \Rightarrow \le n^{4/3 - c}$ edges.

Idea: Many graphs from geometric settings have their edges clustered (nonuniformly distributed)

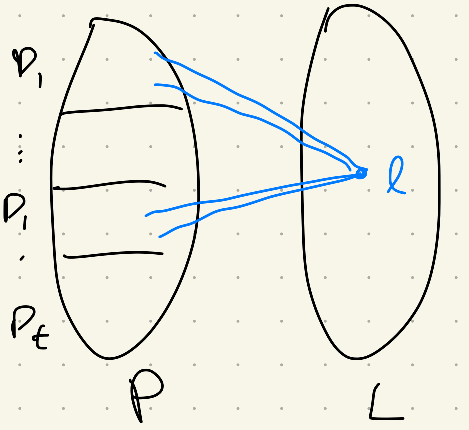
That means we can zoom into some exceptionally dense part. (Matoušek's cutting lem)

We then apply $\overset{\triangle}{}$removal lem.

Lem (Matoušek's cutting lem)

∀ set of n pts $P \subseteq \mathbb{R}^2$, ∀ $2 \le r < n$

∃ partition $P = D_1 \cup D_2 \cup \cdots \cup D_t$, where $r/2 \le t \le r$ and each $\frac{n}{r} \le |D_i| \le \frac{2n}{r}$ and any line passes through pts in $O(\sqrt{r})$ parts



each vx $\ell \in L$ has neighbors is $O(\sqrt{r})$ parts

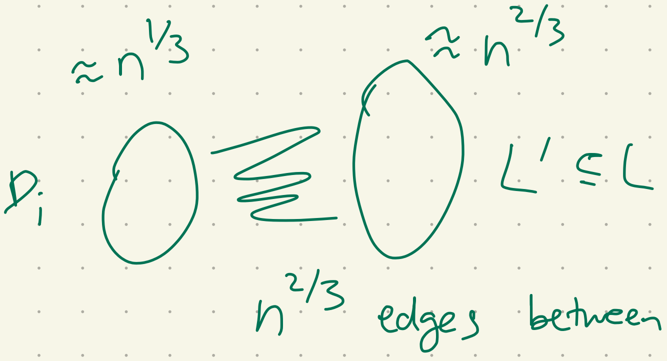
In the pf, suppose

$e(G) = \Omega(n^{4/3})$

- Matoušek cutting

$\xRightarrow{\text{zoom into}}$

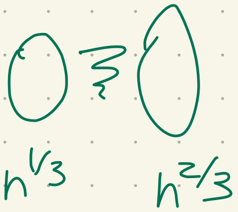
(D_i, L')



exceptionally dense

Original

pure random subsets



$$\mathbb{E}\ \#\text{edges} = n^{4/3} \cdot \frac{n^{1/3}}{n} \cdot \frac{n^{2/3}}{n} = n^{1/3}$$