Lecture 19

- Pf of Kruskal-Katona
- VC-dim

Recall • $\mathcal{F} \subseteq \binom{[n]}{k}$, $\underset{\text{shadow}}{\partial \mathcal{F}} \subseteq \binom{[n]}{k-1}$ all $(k-1)$-sets lying in some $F \in \mathcal{F}$

Kruskal-Katona : given $|\mathcal{F}|$, min. $|\partial \mathcal{F}|$

Thm $\forall\ n_k > n_{k-1} > \cdots > n_s \geq s \geq 1$, $m = \binom{n_k}{k} + \binom{n_{k-1}}{k-1} + \cdots + \binom{n_s}{s}$

$\forall\ \mathcal{F}$ k-set fam. w/ $|\mathcal{F}| = m$ $\implies$ $|\partial \mathcal{F}| \geq \binom{n_k}{k-1} + \binom{n_{k-1}}{k-2} + \cdots + \binom{n_s}{s-1}$

Idea : shifting $\Rightarrow$ some structure + Induction

• Shifting for $2 \leq i \leq n$, for $F \in \mathcal{F}$ define

$$S_i(F) = \begin{cases} F \setminus \{i\} \cup \{1\} & \text{if } i \in F \ \&\ 1 \notin F \text{ and } F \setminus \{i\} \cup \{1\} \notin \mathcal{F} \\ F & \end{cases}$$

Write $S_i(\mathcal{F}) = \{ S_i(F) : F \in \mathcal{F} \}$

Call $\mathcal{F}$ compressed if $S_i(\mathcal{F}) = \mathcal{F}$ for all $2 \leq i \leq n$

Exer Prove that it is a finite process to shift any fam. $\mathcal{F}$ to a compressed one.

Example $\mathcal{F} = \{134, 135, 234, 245\}$

$S_5(\mathcal{F}) = \{134, 135, 234, 124\}$

Rmk Initial segments of colex order are compressed

But not every compressed fam. is initial seg. of colex.

Example $k=2$ $\{13, 15, 17, 19\}$ compressed

Leave the following propositions as exercises.

Prop $\forall \mathcal{F} \subseteq \binom{[n]}{k}$, $\forall\, 2 \le i \le n$, map $\mathcal{F} \longrightarrow S_i(\mathcal{F})$

is injective. In particular, $|\mathcal{F}| = |S_i(\mathcal{F})|$

- Shifting does not increase the size of shadow

Prop $\forall \mathcal{F} \subseteq \binom{[n]}{k}$ $\forall\, 2 \le i \le n$,

$$\Rightarrow \quad \partial\big(S_i(\mathcal{F})\big) \subseteq S_i(\partial \mathcal{F})$$

In particular $\big|\partial(S_i(\mathcal{F}))\big| \le \big|S_i(\partial\mathcal{F})\big| \overset{\text{previous prop.}}{=} |\partial \mathcal{F}|$.

- Given $\mathcal{F} \subseteq \binom{[n]}{k}$, $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_1^c$ where

$$\mathcal{F}_1 = \{F \in \mathcal{F} : 1 \in F\}, \quad \mathcal{F}_1^c = \{F \in \mathcal{F} : 1 \notin F\}$$

remove 1 from all sets in $\mathcal{F}_1$

$(k-1)$-unif

$\mathcal{F}_1^c$ is on ground set $[n]\setminus\{1\}$ good for induction

$$\mathcal{L}_1 = \{F \setminus \{1\} : F \in \mathcal{F}_1\} \quad : \text{link of element } 1.$$

Prop (♡) If $\mathcal{F}$ is compressed, then

(i) $\partial \mathcal{F}_1^c \subseteq \mathcal{L}_1$, and

(ii) $\partial \mathcal{F} = \mathcal{L}_1 \cup \{ E \cup \{1\} : E \in \partial \mathcal{L}_1 \}$

and so $|\partial \mathcal{F}| = |\mathcal{L}_1| + |\partial \mathcal{L}_1|$

Pf (Kruskal-Katona) Double induction on unif k
then on the size of the fam. m

Base $k=1$ trivial as shadow of any 1-unif fam is $\{\emptyset\}$.

Assume $k \geq 2$ and induct on $m = |\mathcal{F}|$

Base $m = 1 = \binom{k}{k} \Rightarrow \mathcal{F}$ has one set, shadow $= \binom{k}{k-1}$

Assume $m \geq 2$. We may assume $\mathcal{F}$ is compressed for otherwise we can keep shifting the fam while keeping its cardinality & not increasing its shadow size.

Claim $|\mathcal{L}_1| \geq \binom{n_k - 1}{k-1} + \binom{n_{k-1} - 1}{k-2} + \dots + \binom{n_s - 1}{s-1}$

Claim $\Rightarrow$ Inductive step

By Prop (♡)(ii) $|\partial \mathcal{F}| = |\mathcal{L}_1| + |\partial \mathcal{L}_1|$

$$\geq \left(\binom{n_k - 1}{k-1} + \binom{n_{k-1} - 1}{k-2} + \cdots + \binom{n_s - 1}{s-1} \right) +$$

$\mathcal{L}_1$ $(k-1)$-unif

I.H.

$|\partial \mathcal{L}_1|$

$$\left(\binom{n_k - 1}{k-2} + \binom{n_{k-1} - 1}{k-3} + \cdots + \binom{n_s - 1}{s-2} \right)$$

$$= \binom{n_k}{k-1} + \cdots + \binom{n_s}{s-1}$$

[Pf of Claim] Suppose not true!

$$\Rightarrow \quad |\mathcal{F}_1^c| = |\mathcal{F}| - |\mathcal{F}_1| = |\mathcal{F}| - |\mathcal{L}_1|$$

$$> \left(\binom{n_k}{k} + \cdots + \binom{n_s}{s} \right) - \left(\binom{n_k - 1}{k-1} + \cdots + \binom{n_s - 1}{s-1} \right)$$

$$= \binom{n_k - 1}{k} + \cdots + \binom{n_s - 1}{s} \quad \cdots\cdots (*)$$

Now, $\mathcal{F}$ is compressed $\Rightarrow |\mathcal{F}_1^c| < |\mathcal{F}|$

IH on $|\mathcal{F}_1^c| \Rightarrow$

$$|\mathcal{L}_1| \geq |\partial \mathcal{F}_1^c| \geq \binom{n_k - 1}{k-1} + \cdots + \binom{n_s - 1}{s-1}$$

Prop (♡) (i)

Issue here: $n_s = s$ possible, then $(*)$ is not of the form in hypo. of K-K. $\boxed{\text{Fix it Exer}}$

Int. k-fam

Size $\le \binom{n-1}{k-1}$

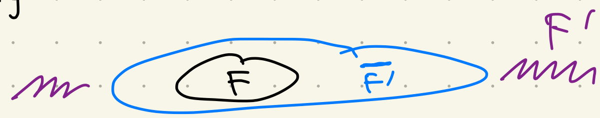
Application K-K $\Rightarrow$ Erdős-Ko-Rado

[Baby K-K] $|\mathcal{F}| \ge \binom{n}{k} \Rightarrow |\partial \mathcal{F}| \ge \binom{n}{k-1}$

Idea $\overline{\mathcal{F}} = \{[n] \setminus F : F \in \mathcal{F}\}$

($\circlearrowleft$) $\forall F \in \mathcal{F}, \overline{F'} \in \overline{\mathcal{F}}$ IMPOSSIBLE: $F \subseteq \overline{F'}$

$\overline{F'} = [n] \setminus F', F' \in \mathcal{F}$



$F' \cap F = \phi$ disj $\lightning$

If $|\mathcal{F}| = |\overline{\mathcal{F}}|$ large

$\Rightarrow$ shadow of $\overline{\mathcal{F}}$ large by K-K

By ($\circlearrowleft$) shadow of $\overline{\mathcal{F}}$ is disjoint from $\mathcal{F}$

$\Rightarrow$ $\mathcal{F}$ small $\lightning$.

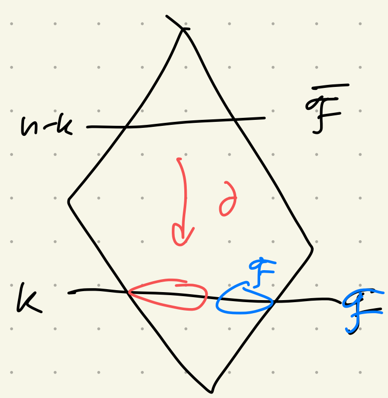
Pf: Suppose $|\mathcal{F}| > \binom{n-1}{k-1}$

$$|\overline{\mathcal{F}}| = |\mathcal{F}| > \binom{n-1}{k-1} = \binom{n-1}{n-k}$$

$(n-k)$-unif

Baby K-K $\Rightarrow$ $|\partial \overline{\mathcal{F}}| \ge \binom{n-1}{n-k-1}$

$$|\partial(\partial \overline{\mathcal{F}})| \ge \binom{n-1}{n-k-2}$$



$\Rightarrow \cdots \quad |\partial^{(n-2k)} \overline{\mathcal{F}}| \ge \binom{n-1}{k}$

k-unif

By ($\circlearrowleft$) $\partial^{(n-2k)} \overline{\mathcal{F}} \cap \mathcal{F} = \phi$

$$\Rightarrow \quad |\mathcal{F}| \leq \binom{n}{k} - \binom{n-1}{k} = \binom{n-1}{k-1}$$ 

§ VC dim & Sauer - Shelah lemma

How can we measure in a sense the complexity of a set system ?

Intuitively, $\mathcal{F} \subseteq 2^{[n]}$ is complex if it 'contain a large copy of a cube $2^{[r]}$' for large r.

In other words, we want to look at how members of $\mathcal{F}$ intersect some subset of the ground set.

distinct intersection limited $\rightarrow$ simple

— " ——— many $\rightarrow$ complex.

Def Given $\mathcal{F} \subseteq 2^{[n]}$, $S \subseteq [n]$, trace of $\mathcal{F}$ on S

$$\mathcal{F}\big|_S = \{ F \cap S : F \in \mathcal{F} \}$$

We say a set S is shattered by $\mathcal{F}$ if

$$\mathcal{F}|_S = 2^S$$

Def The VC-dimension of $\mathcal{F} \subseteq 2^X$ is the maximum size of a set $S \subseteq X$ shattered by $\mathcal{F}$.

could be infinite set

VC-dim $= \infty$ if $\exists$ arbitrarily large shattered set.

Example $X = \mathbb{R}$, $\mathcal{F}$ = collection of all closed intervals $[a, b]$

VC-dim of $\mathcal{F} = 2$