Spring 2024 Extremal Combinatorics

Homework 3

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Due May 2

Please submit **6 out of 8 problems** for grading. You have to choose **3 problems** from No. 1, 2, 3, 4 and **3 problems** from No. 5, 6, 7, 8.

1. Given n points and n lines in the Euclidean plane (no three points are collinear), show that the number of point-line pairs such that the point lie on the line is upper bounded by $O(n^{3/2})$.

2. Let G be an n-vertex graph with $\varepsilon\binom{n}{2}$ edges. Show that there exists a biclique $K_{s,s}$ in G with $s = c\log n$, where $c = c(\varepsilon) > 0$.

3. A *double-pyramid* of size k is a 3-uniform hypergraph consisting of vertices $x, y, z_1, \dots, z_k$ with edge set $\{xz_iz_{i+1}, yz_iz_{i+1}\}_{i\in\mathbb{Z}/k\mathbb{Z}}$. Let $\mathcal{P}$ be the family of double-pyramids of all sizes. Prove that

$$\mathrm{ex}(n, \mathcal{P}) = O(n^{5/2}).$$

4. Show that there exists a constant C such that every graph with average degree at least C contains two cycles of consecutive even lengths.

5. Let h be a positive integer with $h \le \frac{n}{2}$. Show that if $\mathcal{A}$ is an antichain of subsets of an n-set with $|A| \le h$ for any $A \in \mathcal{A}$, then $|\mathcal{A}| \le \binom{n}{h}$.

6. (1) Assume that $k > \frac{n+1}{2}$. Let H be a collection of k-subsets of an n-set S. Let

$$\Delta(H) = \{D \subseteq S : |D| = k-1 \text{ and } D \subseteq A \text{ for some } A \in H\}.$$

Prove that $|\Delta(H)| \ge |H|$.

(2) Use (1) to give a new proof of Sperner's Theorem.

7. Let X be a set of 100 elements. Find the smallest possible n with the following property:

- For any sequence of subsets $A_1, A_2, \dots, A_n$ of X, there exist $1 \le i < j < k \le n$ such that

$$A_i \subseteq A_j \subseteq A_k \quad \text{or} \quad A_i \supseteq A_j \supseteq A_k.$$

8. Let $k < n$ be two positive integers. Let $f(n, k)$ denote the largest value of m for which it is possible to find m chains of $k+1$ distinct subsets of an n-set S such that no member of any chain is a subset of a member of any other chain. Determine $f(n, k)$.