Lecture 9

Previously : $\cdot\ C_4$, $K_{s,t}$ (KST) , $C_{2\ell}$

§ Regularization

A lemma of Erdős - Simonovits allows us to work with almost regular graphs for bipartite Turán problem.

A graph G is **K-almost-regular** if

$$\Delta(G) \leq K \cdot \delta(G)$$

__Lem__ Let $0 < \varepsilon < 1$, $c > 0$ and n be sufficiently large. Let G be an n-vx graph with $e(G) \geq c \cdot n^{1+\varepsilon}$

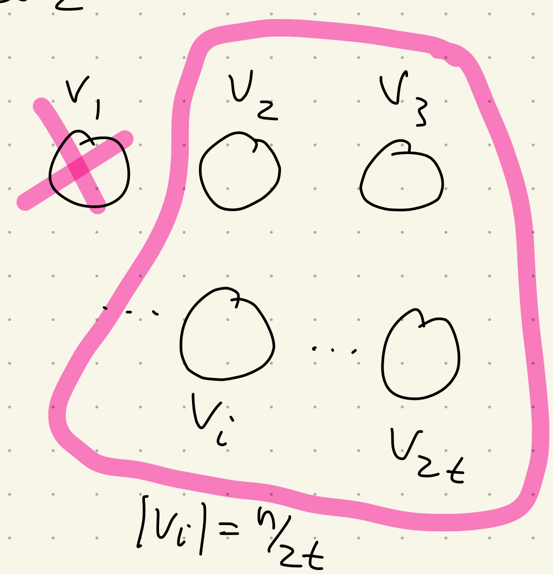
$\Longrightarrow \exists\ G' \subseteq G$ on m vxs , $m \geq n^{\frac{\varepsilon - \varepsilon^2}{4+4\varepsilon}}$ and

- $e(G') \geq \frac{2c}{5} m^{1+\varepsilon}$,
- G' is K-almost-reg , $K = 20 \cdot 2^{1/\varepsilon^2 + 1}$

Pf (Sketch) Exer ⇒ details

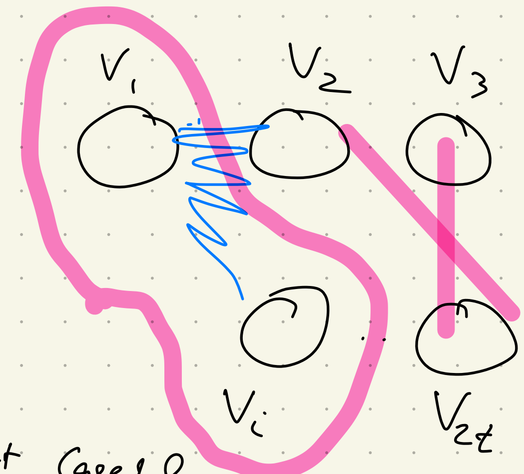
Let $t = K/20 = 2^{1/\varepsilon^2+1}$.

Partition $V(G) = V_1 \cup \dots \cup V_{2t}$ of equal size , where V_1 contains the highest degree vxs



Case 1 $\leq$ half edges incident to V_1

- delete V_1
- repeatedly delete vxs of deg $< \frac{c}{10} n^{\varepsilon}$ (low-deg) $\Longrightarrow G'$ ☺



Case 2 $\geq$ half edges incident to V_1

averaging $\Longrightarrow \exists\, V_i$ s.t.

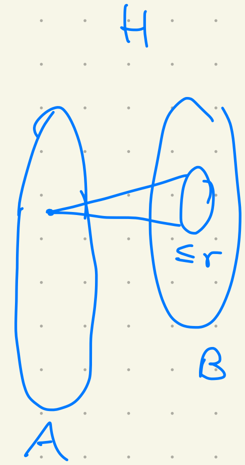
$$e_G(V_1, V_i) \geq \frac{1}{4t} e(G)$$

Work in $G[V_1 \cup V_i]$ & repeat Case 1 & 2 ..... if can be shown that this process must terminates at case 1 for a large graph ☺

Recall KST: $ex(n, K_{r,t}) = O(n^{2-1/r})$, $r \leq t$

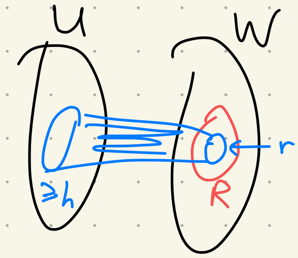
Thm (Füredi, Alon-Krivelevich-Sudakov)

Let H be a bip. gr w/ bipartition $A \cup B$ where each vx in A has deg $\leq r$ in B $\Longrightarrow ex(n, H) = O(n^{2-1/r})$

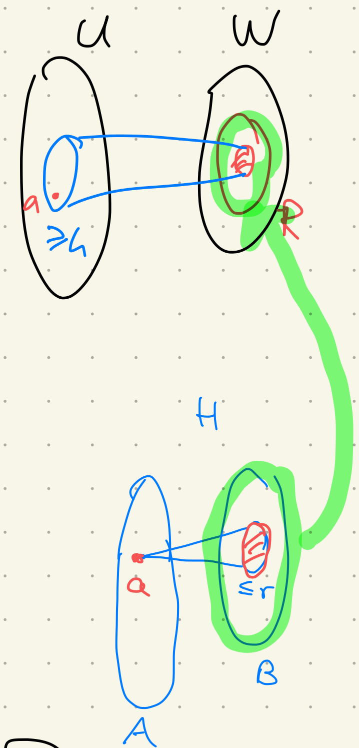


Rmk: extends KST.

Def In a bip. graph w/ $U \cup W$, a subset $R \subset W$ (or $R \subset U$) is called (r,h)-rich if $\forall$ r-set $R' \subset R$ has $\geq h$ common neighbors.



Prop 1 Given a bip. gr G w/ bip. $U \cup W$, if W contains an (r,h)-rich set of size $\geq h$ $\Rightarrow$ G contains any h-vx bip. H w/ max deg r on one side.



Pf:

- Embed $B \xrightarrow{\varphi} R$ rich set
- $\forall a \in A$, map a to an un-used vx in $N_G(\varphi(N_H(a)))$

:)

The next prop. finds a rich subset in an asymmetric bip. gr w/ large degree.

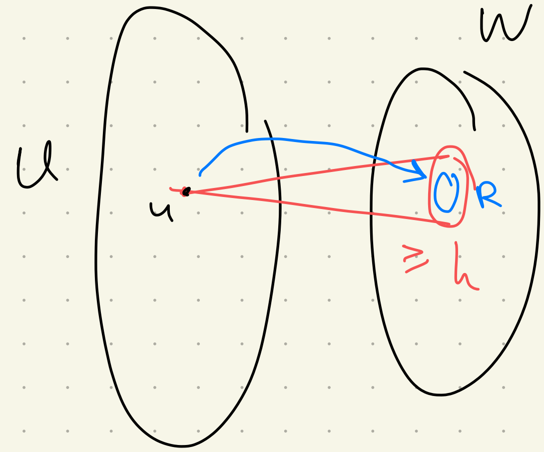
Prop 2 Let H bip. on $A \cup B$, h vxs, vxs in A deg $\leq r$

Let G bip. on $U \cup W$, $\begin{cases} \text{vxs in } U \text{ have deg} \geq h \\ |U| > h\binom{|W|}{r} \end{cases}$

$\Rightarrow$ W contains an (r,h)-rich set of size h

In particular, $H \subseteq G$.

<u>Pf</u>: Shall find such rich set in a neighbrhd of a vx in U.



- Take a maximal partial map

$$\varphi : U \longrightarrow \binom{W}{r} \quad \text{s.t.}$$

(think of vxs in U as colors assigned to r-subsets of W)

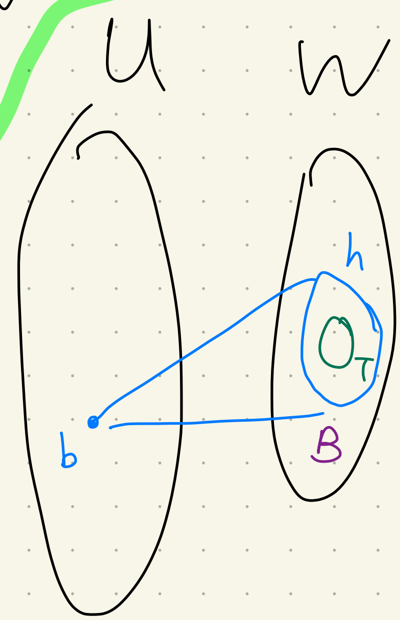
- if $\varphi(u) = R \in \binom{W}{r}$, then $R \subseteq N(u)$,
- $\forall\, R \in \binom{W}{r}$, $\varphi^{-1}(R)$ has size $\leq h$

(i.e. we do not assign more than h colors to any R)

- φ is injective i.e. $|\varphi(u)| \leq 1 \quad \forall\, u \in U$.

As $|U| > h\binom{|W|}{r} \Rightarrow \exists$ a vx $b \in U$ which is not assigned to any r-set in W

<u>Claim</u> An h-set B in $N(b)$ is (r,h)-rich.



<u>Pf</u> NTS $\forall$ r-set $T \subseteq B$ has $\geq h$ common neighbors

Note $|\varphi^{-1}(T)| = h$

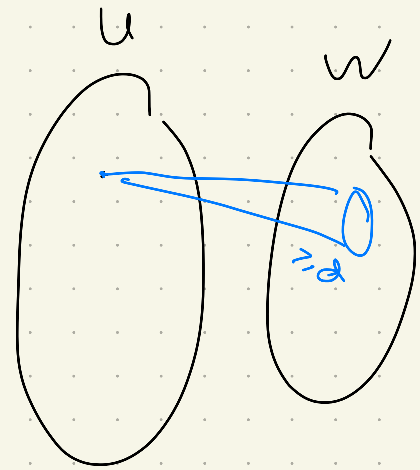
we have h colors assigned to T.

for o.w. letting $\varphi(b) = T$ $\lightning$ maximality of φ.

<u>Thm 3</u> (Random zooming , Gil-Fernandez - Hyde - Liu - Pikhurko - Wu)

Let $d \geq \max\{40, 2h\}$ and G be a bip. gr on $U \cup W$ s.t. every vx in U has deg $\geq d$ in W. If

$$(\heartsuit) \ldots \quad \frac{1}{2|W|}\left(\frac{|U|}{4h}\right)^{1/r}\frac{d}{2} \geq \max\{20, h\}$$



$\Longrightarrow$ G contains any h-vx bip. H w/ max deg r on one part.

<u>Exer:</u> Thm 3 $\Longrightarrow$ F-A-K-S.

<u>Idea</u> : By <u>Prop 2</u> , suffices to find $U' \subset U, W' \subset W$ inducing asymmetric bip. w/ large deg on U' side

To find such U' & W', we randomly zoom into a subset $W' \subset W$.

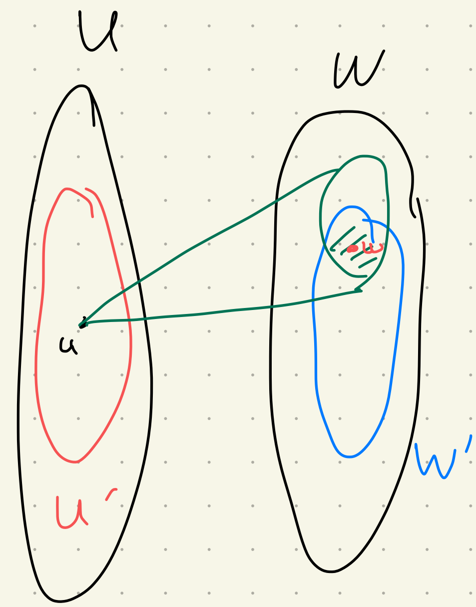
<u>Pf</u> : • Let $p = \frac{1}{2|W|} \cdot \left(\frac{|U|}{4h}\right)^{1/r}$

If $|U| > 4h(2|W|)^r$, replace it w/ a subset of size exactly $4h(2|W|)^r \Longrightarrow (\heartsuit)$ holds & $p \leq 1$

- Let $W' \subset W$ be a p-random subset of W where each vx in W is chosen w/ prob. p indep. of others.

- For $u \in U$, $X_u := d_G(u, W')$

$$= \sum_{w \in N(u)} \mathbb{1}_{\{w \in W'\}}$$



Sum of indep. Bernoulli r.v.

$$\mathbb{E} X_u = p \cdot d(u) \geq p \cdot d$$

$\left\{ \begin{array}{l} \text{low-tail Chernoff} \\ pd/12 \geq 2 \end{array} \right. \Rightarrow$

$$\Pr\left(X_u < pd/2\right) \leq e^{-pd/12} \leq 1/4$$

Let $U' = \{u \in U : X_u \geq pd/2\}$

<u>Claim</u> $\Pr\left(|W'| > 2p|W|\right) + \Pr\left(|U'| < |U|/4\right) < 1$

- Claim $\Rightarrow$ with positive probability none of these two events happen. That is: $\left\{ \begin{array}{l} |W'| \leq 2p|W| \\ |U'| \geq |U|/4 \end{array} \right.$

$$+ (\heartsuit) \implies |U'| > h\left(\frac{|W'|}{r}\right)$$

- Choice of U' $\Rightarrow$ deg to $W' \geq pd/2 \geq h$

<u>Pf of Claim</u> (i) $\Pr\left(|U'| < |U|/4\right) = q \leq 1/2$

Suppose $q > 1/2$

$$\boxed{\Pr\left(X_u < pd/2\right) \leq e^{-pd/12} \leq 1/4}$$

$$\mathbb{E}|U'| = |U| \cdot \Pr\left(X_u \geq pd/2\right) \geq \frac{3|U|}{4}$$

$$\mathbb{E}|U'| \leq q \cdot \frac{|U|}{4} + (1-q)|U| \leq \frac{5|U|}{8} \quad \text{↯}.$$

(ii) $\Pr\left(|W'| > 2p|W|\right) \leq 1/4$.

As $|W| \geq d \Rightarrow \mathbb{E}|W'| = p|W| \geq pd \geq 40$

upper tail Chernoff $\Rightarrow$

$$\Pr\left(|W'| > 2p|W|\right) \leq e^{-\frac{p|W|}{3}} \leq 1/4$$