- Applications of Dilworth's thm

- Ramsey for intersection graphs of planar convex sets.

Lecture 14

Last time: · Poset

Duality thm   Dilworth's thm   max antichain $=$ min chain decomposition

Pf 1. Induction on $|P|$

Exer Pf 2   Use symmetric chain decomposition

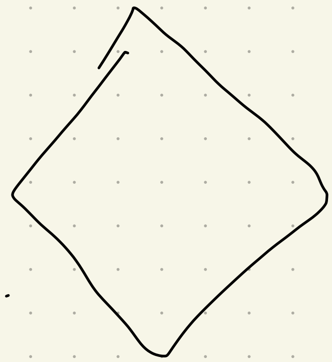
Use induction to find

Def A chain in $(2^{[n]}, \subseteq)$ is symmetric

$$C = (A_k, A_{k+1}, \cdots, A_{n-k})$$

for some $k = 0, 1, \cdots, n$ s.t.

$$|A_i| = i \quad \forall i \in \{k, k+1, \ldots, n-k\}.$$



Exer Pf 3

Thm Dilworth's Thm $\iff$ Kőnig's thm.

Def Given a graph G, the vertex cover # of G, denoted by $\tau(G)$, is the min. # vxs to cover all edges

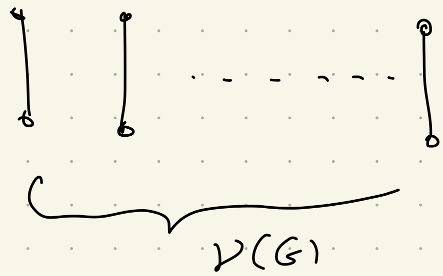
$T \subseteq V(G)$: $T \cap V(e) \neq \emptyset$ $\forall e \in E(G)$

Rmk Transversal # of G.

Obs $\forall G$, vertex cover # $\geq$ matching #

$$\tau(G) \geq \nu(G)$$

pf



- The gap could be arb. large.

Exer Think of an example

Thm (König) $\forall$ bip. gr G, $\tau(G) = \nu(G)$

§ Applications of Dilworth's thm.

Ramsey theory : Within chaos, there is order.
philosophy

- $\forall$ 2-edge-coloring of K_n, if n large $\Rightarrow$ $\exists$ large monochromatic clique

arbitrary structure

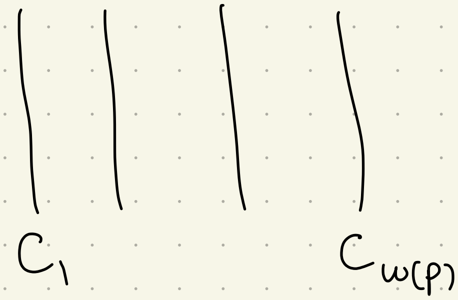
$\Theta(\log n)$

large homogeneous substructure

Cor $\forall$ poset $(P, <)$, $|P| = n$, $h(P) \cdot w(P) \geq n$

largest chain, largest antichain

__Pf__ Dilworth $\exists$ a chain decomp. of size $w(P)$



$$n = \left| \bigcup C_i \right| \le w(P) \cdot h(P).$$ ☺

In other words

__Def__ Comparability gr G for a poset $(P, <)$

$V(G) = P$

$x \sim_G y \iff x \ \& \ y$ comparable $\left(\begin{matrix} x < y \\ \text{or} \\ y < x \end{matrix} \right)$

G: clique $\iff$ chain in P
indep set $\iff$ antichain

Cor above says
$\Downarrow$
__Cor__ $\forall$ comparability gr on n vxs has an indep or a clique of size $\geq \sqrt{n}$.

__Rmk__ Dilworth's thm illustrates that more structure $\Rightarrow$ better Ramsey.

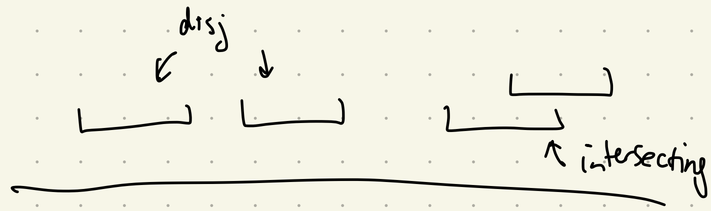
- In many geometrically structure, we see much better Ramsey statement.
  We shall see some examples for which we associate poset structure, hence better Ramsey.

§ Interval graphs

$a \leq \ \leq b$ (closed)

Consider a collection of intervals $\mathcal{I} = \{I_1, I_2 .., I_n\}$

Associate a gr G



$V(G) = \mathcal{I}$

two vxs adj $\iff$ the corresponding intervals are intersecting

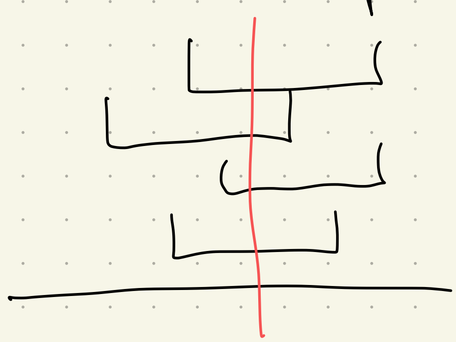
Def Intersection gr of $\mathcal{I}$

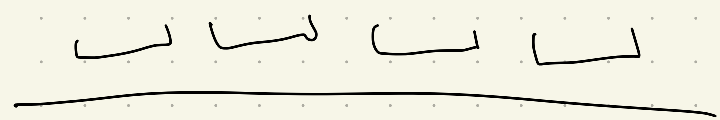
Disjointness gr = complement of intersection gr.

- In intersection gr G

Clique = subcollection intervals pairwise intersecting

indep set = pairwise disjoint intervals





Prop (Exer) Any collection of pairwise intersecting intervals $\{I_1, \dots, I_n\}$ is intersecting, i.e. $I_p \cap I_q \neq \emptyset \ \forall p, q \in [n]$

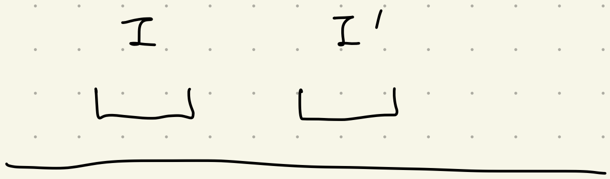
$$\Rightarrow \bigcap_{r \in [n]} I_r \neq \emptyset$$

Thm $\forall\ \mathcal{I} = \{I_1, \dots, I_n\}$ intervals, $\exists \geq n^{1/2}$ of them that are

- either intersecting
- OR disjoint.

Idea Classify pairs of intervals by a bdd # of poset, then apply Dilworth iteratively.

Pf



① $I < I'$ if disjoint & $\max I < \min I'$

transitive

$(P, <)$ on $\mathcal{I}$

By Dilworth $\Rightarrow$ either chain $\geq \sqrt{n} \to$ disjoint

or antichain $\geq \sqrt{n}$

$\sqrt{n}$ intervals that are pairwise intersecting

$\overset{\text{Prop}}{\Longrightarrow}$ they are intersecting.

Intervals are convex sets in $\mathbb{R}^1$.

Let's consider those in $\mathbb{R}^2$

Q: Given n convex sets in the plane $\mathbb{R}^2$, how many of them we can find that are pairwise disjoint or pairwise intersecting ?

$\iff$ $\forall$ intersection gr of n convex sets how large an indep set or clique we can guarantee ?

Thm (Larman - Matoušek - Pach - Töröcsik)

$\forall$ n conv sets in the plane $\mathcal{C}$

$\Rightarrow$ $\geq n^{1/5}$ many of them that are pairwise intersecting or pairwise disjoint

There are arrangement with no pairwise intersecting or pairwise disjoint planar conv sets of size $n^{\log 2/\log 5} \leq n^{0.431}$.

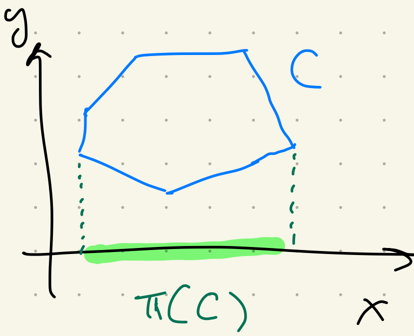
Rmk Cannot hope for any $\omega(\log n)$ bound for intersection gr of conv sets in $\mathbb{R}^3$

Because by a result of Tietze, every gr can be realized as intersection gr of conv sets in $\mathbb{R}^3$.

pf Shall use 4 posets to classify pairs of planar conv sets.

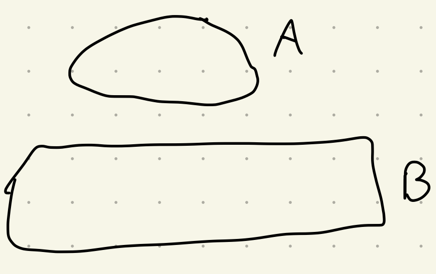
In all 4 poset, two planar conv sets in $\mathcal{C}$ are comparable only if they are disjoint.

$\forall\, C \in \mathcal{C}$, write $\pi(C)$ the projection of C onto x-axis



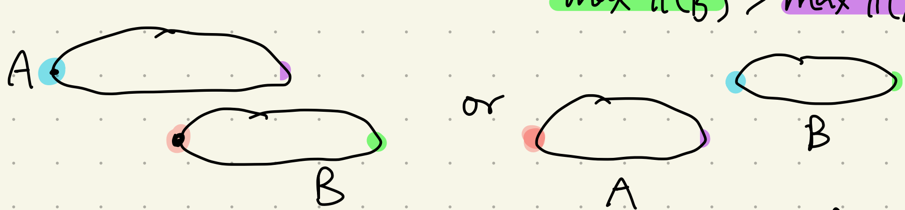
1) $A <_1 B$ $\begin{cases} \cdot\ A \cap B = \emptyset \ \& \\ \cdot\ \pi(A) \subseteq \pi(B) \ \& \\ \cdot\ A \text{ lies below } B \end{cases}$

2) $A <_2 B$ $\begin{cases} \cdot\ A \cap B = \emptyset \ \& \\ \cdot\ \pi(A) \subseteq \pi(B) \ \& \\ \cdot\ A \text{ lies above } B \end{cases}$



3) $A <_3 B$ $\to$ if (i) $\min \pi(B) > \min \pi(A)$ left endpt of $\pi(B)$ to the right of left endpt of $\pi(A)$

(ii) right endpt of $\pi(B)$ to the right of right endpt of $\pi(A)$ $\max \pi(B) > \max \pi(A)$



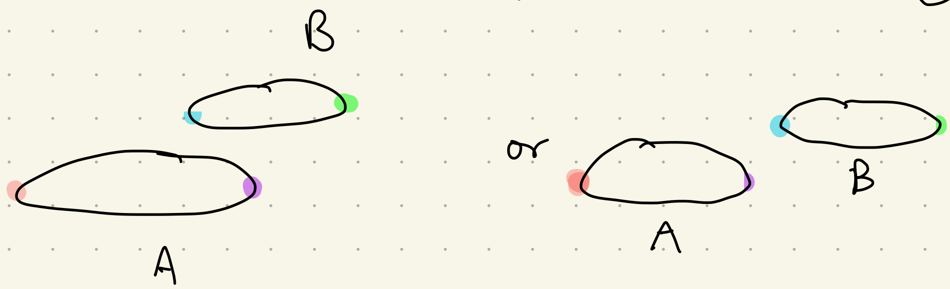
(iii) the part $\pi(A)$ & $\pi(B)$ overlap

A lies above B.

4) $A <_4 B$: if (i) (ii) same as 3)

but for (iii) the overlap part

A lies below B.



Exer 1 $\forall$ two disjoint conv sets A, B in $\mathbb{R}^2$

$A <_i B$ for some $i \in [4]$

antichain in all 4 posets $\Rightarrow$ pairwise intersecting.

Exer 2 $<_i$ is transitive $\forall\, i \in [4]$

Apply Dilworth's thm on $(P_1, <_1)$

$\Rightarrow$ • either chain (pairwise disjoint ☺)

size $n^{1/5}$

• OR antichain A_1 of size $n^{4/5}$

Dilworth on A_1 w.r.t. $<_2$

$\Rightarrow$ • either chain (pairwise disjoint ☺)

size $n^{1/5}$

• OR antichain A_2 of size $n^{3/5}$

For $i \in \{1, 2, 3\}$

Dilworth on A_i w.r.t. $<_{i+1}$

$\Rightarrow$ • either chain (pairwise disjoint) size $n^{1/5}$

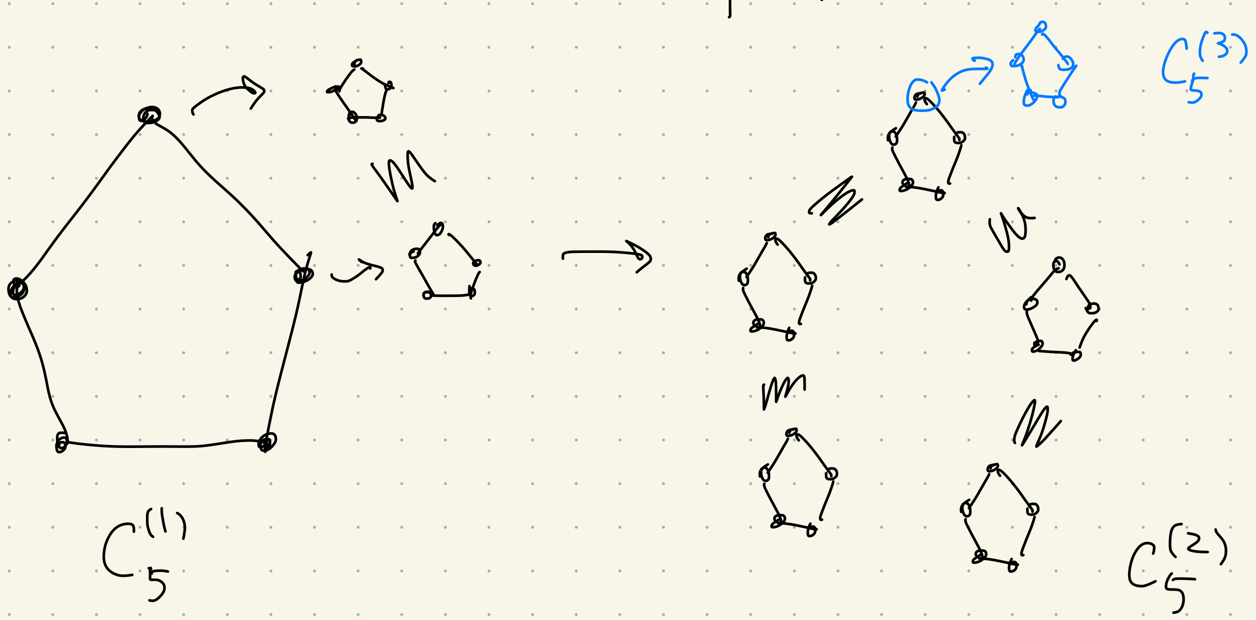
• OR antichain A_{i+1} of size $n^{\frac{4-i}{5}}$

At the end $i=3 \Rightarrow A_4$ of size $n^{1/5}$

antichain in all 4 posets $\Rightarrow$ pairwise intersecting.

For the upper bd $n^{\log 2/\log 5}$:

Iterative blowup of C_5 can be realized as intersection gr of planar conv sets



$$C_5^{(t)} \begin{cases} \#\text{ vxs} = 5^t \longrightarrow n \\ \text{largest clique, indep sets} = 2^t \longrightarrow n^{\log 2/\log 5} \end{cases}$$