Lecture 13

Recall Bollobás set-pairs ineq $A_1, B_1, \ldots, A_m, B_m$

$$(*)\begin{cases} A_i \cap B_i = \emptyset & \forall i \\ A_i \cap B_j \neq \emptyset & \forall i \neq j \end{cases} \Rightarrow \sum_{i \in [m]} \frac{1}{\binom{|A_i|+|B_i|}{|A_i|}} \leq 1$$

Open problem $(A_i, B_i)_{i \in [m]}$ is (a,b)-unif ISP

if $|A_i| = a$, $|B_i| = b$ & it satisfies $(*)$

Q: What is the max ground set of an (a,b)-unif ISP?

Tuza 80s

- Application of Sperner's thm: $\mathcal{A}$ antichain in $2^{[n]}$
$\Rightarrow |\mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor}$

Littlewood-Offord problem bounds the atom probability of Rademacher sum. (useful in Random matrix theory, bounding singularity prob. of random matrices)

- Rademacher r.v.

$$= \begin{cases} 1 & \ldots \frac{1}{2} \\ -1 & \ldots \frac{1}{2} \end{cases}$$

Thm (Erdős 1945) Let $a_1, \ldots, a_n \in \mathbb{Z} \setminus \{0\}$ be non-zero integers and $x_1, \ldots, x_n$ be iid Rademacher random var.

$$\Rightarrow \sup_{z \in \mathbb{Z}} \Pr\left(\sum_{i \in [n]} a_i x_i = z\right) \leq \frac{\binom{n}{\lfloor n/2 \rfloor}}{2^n} = \Theta\left(\frac{1}{\sqrt{n}}\right)$$

- Stirling formula $\Rightarrow$ $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$

Idea link a random sum to a subset in $[n]$.

Pf
- 2^n choices for the random signs $x_1, \dots, x_n$.

Fix $z \in \mathbb{Z}$ and let $S \subseteq \{-1, 1\}^n$ be the set of all choices $\Rightarrow$ $\sum_{i \in [n]} a_i x_i = z$.

$$\Rightarrow \Pr\left(\sum a_i x_i = z\right) = \frac{|S|}{2^n}$$

STS $|S| \le \binom{n}{\lfloor n/2 \rfloor}$

By Sperner, $\boxed{\text{STS}}$: $S \subseteq \{-1, 1\}^n$ is antichain

- By symm. of Rademacher r.v., may assume that all $a_1, \dots, a_n$ are positive.

Identify S w/ a subset of $[n]$, i.e.

$$\left(\begin{array}{lll} \text{if} & x_i = 1, & i \in S \\ & x_i = -1, & i \notin S. \end{array}\right)$$

Claim S is an antichain.

Suppose $\exists$ two sets $Y \subsetneq Y'$

$(y_1, \dots, y_n) \qquad (y_1', \dots, y_n')$

$-1 \quad 1 \qquad \pm 1 \quad 1$

$$\Rightarrow \underset{Y, Y' \in S}{0 =} \sum a_i y_i - \sum a_i y_i' = 2 \sum_{i \in Y' \setminus Y} a_i > 0$$

non empty

$\lightning$

Rmk: Bound is Optimal. Consider $a_i = a$, $z = 0$, n even

$$\Pr\left(\sum_{i\in[n]} a_i x_i = 0\right) = \Pr\left(\sum_{i\in[n]} x_i \cdot \cancel{a} = 0\right) = \frac{\binom{n}{n/2}}{2^n}$$

happens when $n/2$ $+1$

$n/2$ -1

§ Poset

Poset: partial order set $(P, <_P)$

$(P, <)$

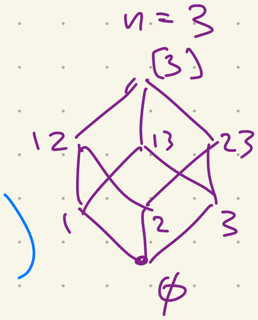
is a set P with a binary relation $<$

that is

- reflexive $(\forall x, \ x < x)$
- antisymmetric $(x < y, y < x \Rightarrow x = y)$
- transitive $(x < y, y < z \Rightarrow x < z)$

· Compare this to $1 < 2 < 3$

$(\mathbb{N}, <)$ totally ordered . . . . . .

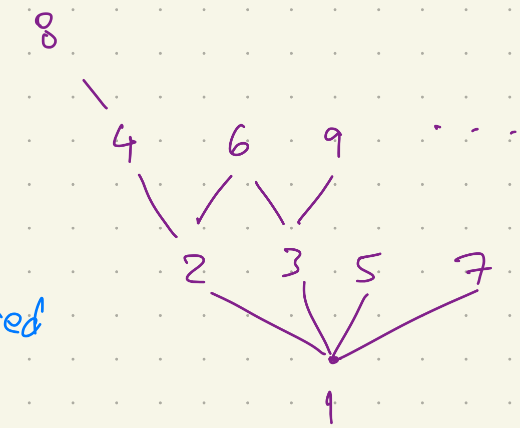


Examples 1) Boolean poset $(2^{[n]}, \subseteq)$

Subset containment

2) Divisibility lattice $(\mathbb{N}, |)$

$P = \mathbb{N}$, $a|b \iff a <_P b$



3) vector sp. V

set of all subspaces of V ordered by subsp. containment.

Some important invariants for posets $(P, <)$

Def Two element a, b are comparable if $a < b$ or $b < a$

o.w. incomparable.

A subset $A \subseteq P$ is an antichain if elements of A are pairwise incomparable.

largest size = width of poset

A subset $C \subseteq P$ is a chain if elements of C are pairwise comparable

largest chain = height of poset.

$C = \{x_1, \dots, x_r\}$

Rmk $\forall C$ chain, Transitivity $\Rightarrow$ totally ordered $x_1 < x_2 < \dots < x_r$

Def $(P, <)$ Poset, a chain / antichain decomposition of P: $P = C_1 \cup \cdots \cup C_t$

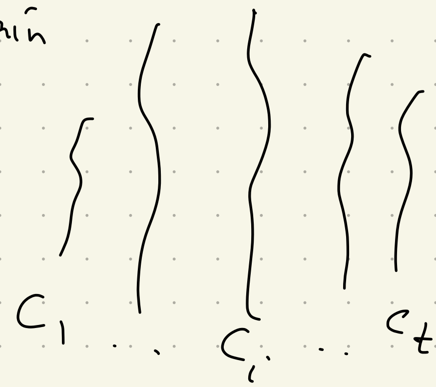
$P = A_1 \cup \cdots \cup A_s$

Obs $\forall$ chain decomposition $P = C_1 \cup \cdots \cup C_t$

$\forall$ antichain $A \subseteq P$

size chain decomp. $\geq$ size antichain

$$t \geq |A|.$$



Since elements of A have to lie in distinct chains in any chain decomp.

Dilworth's thm is a duality statement infering that $=$ holds for min chain decomp & max antichain.

Thm (Dilworth) $\forall$ poset $(P, <)$

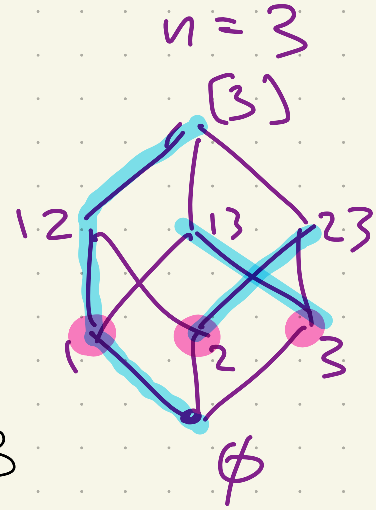
max antichain $=$ min chain decomp.

$=$ width of P

By Dilworth, if we can find a chain decomp & an antichain of the same

$\Rightarrow$ width of poset

Ex



width = 3

<u>Example</u>

1.) Width of Boolean poset

$= \binom{n}{\lfloor n/2 \rfloor}$ by Sperner's thm

$\Rightarrow$ min chain decomp. of $(2^{[n]}, \subseteq)$ is of size $\binom{n}{\lfloor n/2 \rfloor}$

2) Width of $([2n], \text{divisibility})$

antichain of size n : $\{n+1, \dots, 2n\}$

max

$\|$

min

Chain decomp. of size n, $\forall x$ odd $\to$ $x \cdot 2^i$
$i = 0, 1, 2, \dots$

	4	$2^2 \cdot 3$	$2^2 \cdot 5$	$2^2 \cdot 7$
	$\vert$	$\vert$	$\vert$	$\vert$
	2	$2 \cdot 3$	$2 \cdot 5$	$2 \cdot 7$
	$\vert$	$\vert$	$\vert$	$\vert$
odd numbers	1	3	5	7 .....

A cor. of Dilworth's thm is a Ramsey type statement, illustrating a phenomenon that

more structure $\Rightarrow$ better quantitative Ramsey

$\forall (P, <)$ width $= w$

$\Rightarrow$ $h \cdot w \geq \sum_{i=1}^{w} |C_i| = n = |P|$

Cor $\forall$ poset on n elements

$\Rightarrow$ $\exists$ a chain or an antichain of size $\geq \sqrt{n}$

Rmk Color $E(K_n)$ by comparability of a poset

(red $\Leftrightarrow$ incomp, blue $\Leftrightarrow$ comp.) $\Rightarrow$ clique = chain

indep set = antichain

$\Rightarrow$ $\sqrt{n}$ size monochromatic clique.

- Standard Ramsey $\frac{1}{2} \log_2 n \leq$ largest mono clique $\leq 2 \log_2 n$

Exer Prove Erdős-Szekeres on monotone seq using Dilworth.

Dual of Dilworth's thm : Mirsky's thm

$\forall$ chain $\leq$ $\forall$ antichain decomposition

<u>Mirsky Thm</u> Max chain = min antichain decomp.

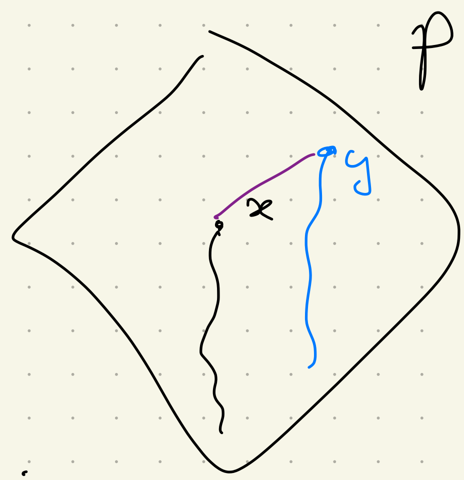
<u>Idea</u> : consider level sets of height function.

<u>Pf</u> : Define $h : P \longrightarrow \mathbb{N}$

$\forall \; x \in P \longmapsto h(x) =$ size of largest chain w./ max $= x$

<u>Claim</u> $\forall \, i \in \mathbb{N}$

Level sets $h^{-1}(i)$ are antichains



$\Rightarrow$ as # level sets

= height of P

= size of largest chain

<u>Pf</u> If not $k = h(x) = h(y)$

But $x < y$, $\Rightarrow$ consider the chain

$\{y\} \cup C_x$ is of size $k+1$

C_x = longest chain w./ max $= x$

$\Rightarrow h(y) \geq k+1$ ↯

Pf (Dilworth's thm by Induction on $|P|$)

Base case $|P| = 1$ ✓

Inductive Step Let a be a maximal element of P. ($\nexists\, b$ s.t. $a < b$)

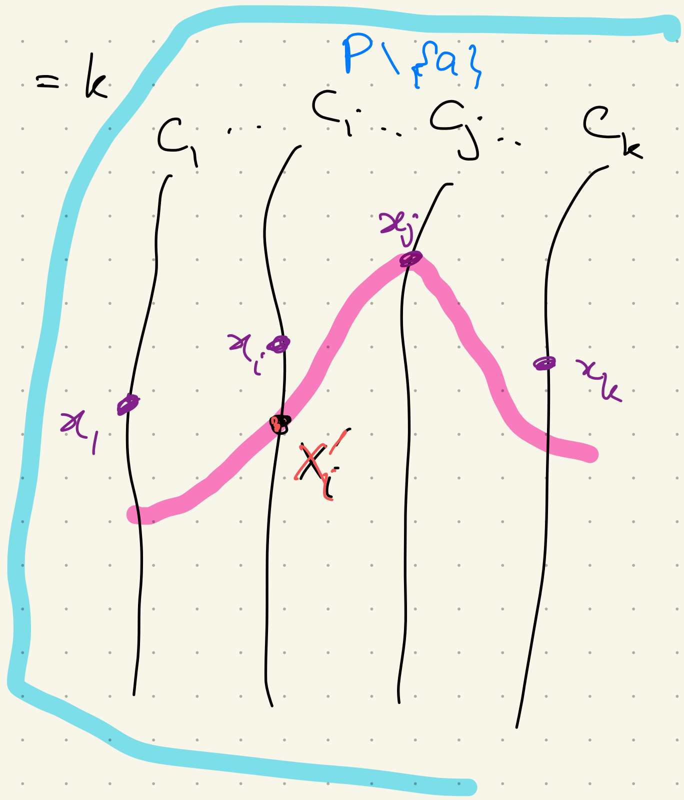
- IH $\Rightarrow$ $\exists$ chain decomp. $P' = P \setminus \{a\} = C_1 \cup \cdots \cup C_k$

  & largest antichain of $P' = k = \text{width}(P')$

- If $\text{width}(P) = k+1$, done as

  $P = C_1 \cup \cdots \cup C_k \cup \{a\}$ is a chain decomp.

  May assume $\text{width}(P) = k$



- Define $x_i \in C_i$ as the max. element that lies in an max antichain (of size k)

Well-defined as no size k antichain can miss a chain in $C_1, \ldots, C_k$

<u>Claim</u> $\{x_1, \dots, x_k\}$ is an antichain.

<u>Pf</u>. Suppose $x_i < x_j$

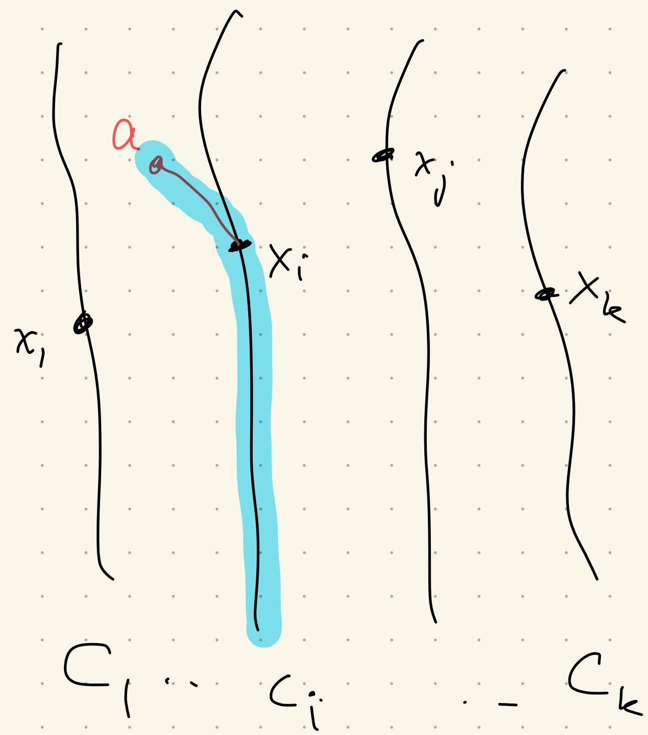
Let A' be a size-k antichain $\ni x_j$

$X_i' = A' \cap C_i$

Choice of $x_i \Rightarrow x_i' < x_i < x_j$ ↯

But $x_i', x_j \in A'$ not comparable. ☺

- If a incomparable to any of $x_1, \dots, x_k \Rightarrow$ width(P) $= k+1$ ✓



- So $a > x_i$ for some $i \in [k]$

$C = \{a\} \cup \{x \in C_i : x < x_i\}$

- Choice of x_i

$\Rightarrow$ Size of antichain in $P - C \leq k-1$

IH on $P - C \Rightarrow P - C = C_1' \cup \dots \cup C_{k-1}'$ ☺