Announcements:

1) • April 16 lecture (Tuesday)
   ⟹ April 15 (Monday)

2) • HW 50% / exam 50%

3) Summer Research program

   July & August.

Lecture 12

Recall $ex(n, C_8) = O(n^{1+1/4})$

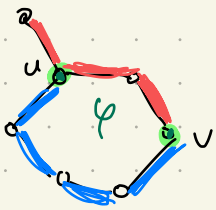
Shown: $\hom(\textcircled{6}, G) \approx \hom(\textcircled{8}, G)$

$= \Omega(\cdot)$

Compare $\textcircled{4}$ & $\textcircled{6}$

Claim $\textcircled{6}^2 \leq \textcircled{8} \cdot \textcircled{4}$

$$\left(\sum a_i b_i\right)^2 \leq \left(\sum a_i^2\right)\left(\sum b_i^2\right)$$



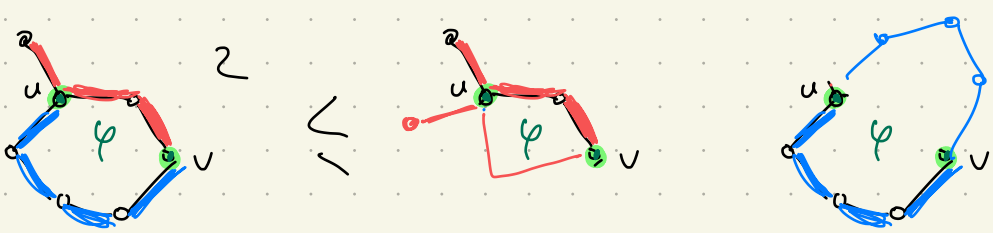
φ: all hom of $u \cdots v \to G$

a_φ: # extensions of φ to P_4-hom $u - \cdot - \cdot - v$ (φ)

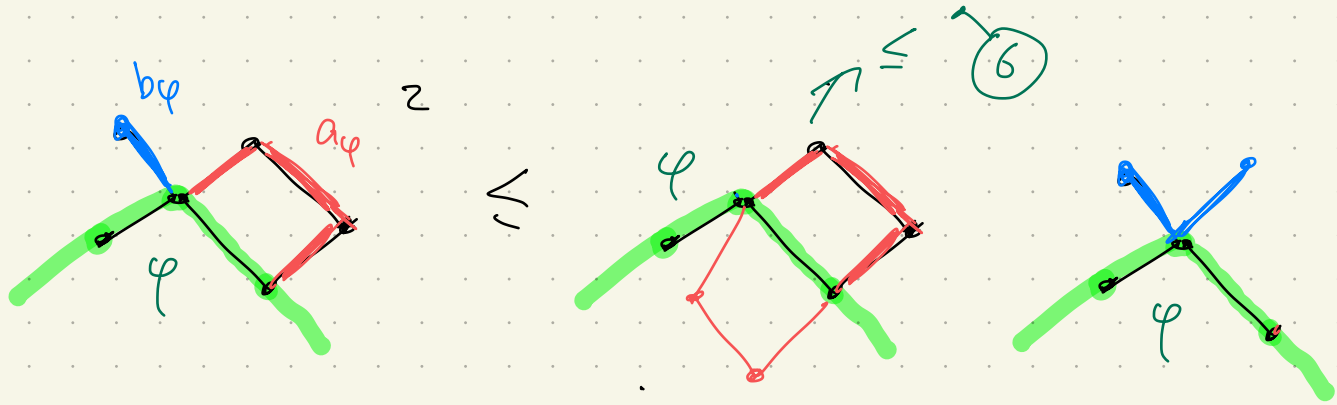
b_φ = # extensions of φ : P_5-hom w/ endpts u & v.

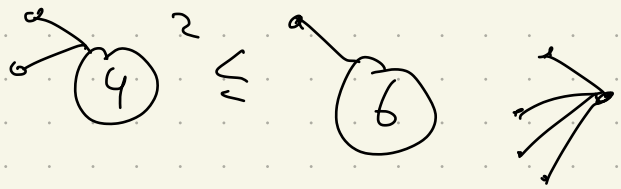
$$\left(\sum_\varphi a_\varphi b_\varphi\right)^2 \leq \left(\sum a_\varphi^2\right)\left(\sum b_\varphi^2\right)$$

$\textcircled{6}^2 \leq \textcircled{4} \cdot \textcircled{8}$

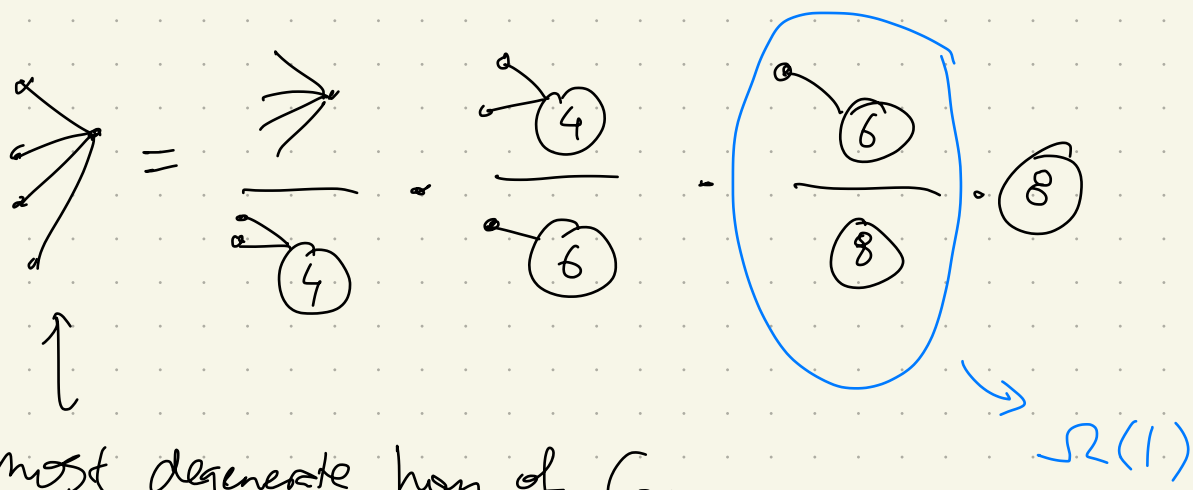


$$\frac{\textcircled{}}{\textcircled{4}} \geq \frac{\textcircled{4}}{\textcircled{6}} \geq \frac{\textcircled{6}}{\textcircled{8}}$$





Final contradiction using Sidorenko



most degenerate hom of G_{uk}

$$n^5 p^4 \approx \quad \geq \Omega(1) \cdot 8 \overset{\text{Sidorenko}}{\geq} n^8 p^8$$

$$p \le n^{-3/4}$$

$\boxed{:)}$

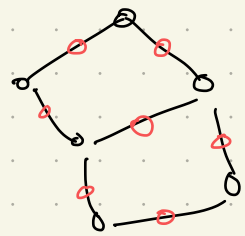
Conj (Rational exponent)

$\forall$ rational $r \in (1,2)$, $\exists$ bip. H s.t.

$$ex(n, H) = \Theta(n^r)$$

Known Bukh-Conlon: True if forbidding a finite family of bip. gr.

Def 1-subdivision



Conj (Subd. conj Kang-Kim-Liu)

If $ex(n, H) = O(n^{1+\alpha})$ for some $0 < \alpha \leq 1$

$$\Longrightarrow \quad ex(n, \underline{sub(H)}) = O(n^{1+\alpha/2})$$

1-subd. of H

Rmk Subd. conj, if true, implies Rational exponent conj.

§ Extremal Set Theory

Extremal problems for family of sets with various intersection pattern.

- containment $A \subseteq B$ (Sperner)
- disjointness $A \cap B = \emptyset$
- intersecting $A \cap B \neq \emptyset$ (Erdős-Ko-Rado)

<u>Recall</u>

- odd/even town: restriction on parity of size of pairwise intersection
- Sperner: $\forall$ antichain $\mathcal{A} \subseteq 2^{[n]}$

  $\forall A, A' \in \mathcal{A}$, $A \not\subseteq A'$

  $$|\mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor}$$
- LYM ineq $\forall$ antichain $\mathcal{A} \subseteq 2^{[n]}$

$$\Rightarrow \sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{|A|}} \leq 1$$

<u>Thm</u> (Bollobás set-pairs ineq) Given sets $A_1, \cdots, A_m, B_1, \cdots, B_m$

if $\begin{cases} A_i \cap B_i = \emptyset & \forall i \in [m] \\ A_i \cap B_j \neq \emptyset & \forall i \neq j \end{cases} \Rightarrow \sum_{i \in [m]} \frac{1}{\binom{|A_i| + |B_i|}{|A_i|}} \leq 1$

In particular, if $\forall i \in [m]$, $|A_i| = a$, $|B_i| = b$

$$\Rightarrow \quad m \le \binom{a+b}{a}$$

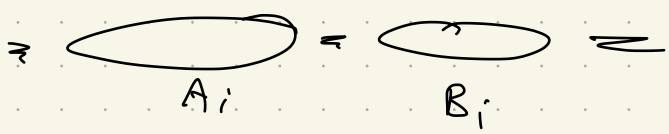
<u>Rmk</u> Note that the size of ground set (of A_i, B_i) do not play a role.

<u>Pf</u>. Assume ground set $= [n]$

- Consider a uniform random permutation σ of $[n]$

  event P_i : in σ, A_i precedes B_i

  $\max A_i < \min B_i$



- P_i depends only on relative positions of $A_i \cup B_i$ $\longrightarrow \sigma$

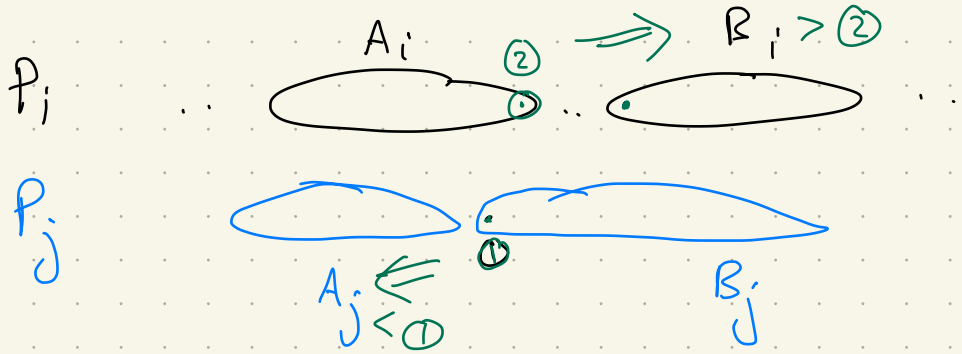
$$\Pr(P_i) = \frac{|A_i|!\ |B_i|!}{(|A_i| + |B_i|)!} = \frac{1}{\binom{|A_i|+|B_i|}{|A_i|}}$$

Suffices to show P_i, $i \in [m]$, are disjoint events

then $\sum \Pr(P_i) \le 1$ ☺

<u>NTS</u> if P_i occur, P_j cannot happen $j \ne i$

Suppose both P_i, P_j occur.



- $A_i \cap B_j \neq \emptyset \Rightarrow \underset{①}{\min B_j} < \underset{②}{\max A_i} \overset{P_i}{<} \min B_i$

$\max A_j \underset{P_j}{<}$

$\Rightarrow \max A_j < \min B_i$ ↯ as $A_j \cap B_i \neq \emptyset$.

<u>Exer</u> Find an example showing tightness of Bollobás set-pairs ineq.

Skewed ver.

<u>Thm</u> (Bollobás set-pairs ineq) Given sets $A_1, \cdots, A_m, B_1, \cdots, B_m$, $|A_i| \leq a$, $|B_i| \leq b$

if $\begin{cases} A_i \cap B_i = \emptyset & \forall i \in [m] \\ A_i \cap B_j \neq \emptyset & \forall i < j \end{cases} \Rightarrow m \leq \binom{a+b}{a}$

<u>Rmk</u> We only have an alg. pf of this skewed by Lovász, Alon-Kalai. Finding a combinatorial pf would be interesting.

- V vector space over a field K

The k-th exterior power of V, denoted by, $\Lambda^k V$ is a vector sp. spanned by elements of the form

$$x_1 \wedge x_2 \wedge \dots \wedge x_k, \quad x_i \in V, \forall i \in [k]$$

s.t. $x_1 \wedge x_2 \wedge \dots \wedge x_k = 0$ if $x_i = x_j$ for some distinct $i, j \in [k]$.

- If $\{e_1, \dots, e_n\}$ is a basis of V,

then a basis for $\Lambda^k V = \{ e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_k} : 1 \leq i_1 < i_2 < \dots < i_k \leq k \}$

$\dim V = n \Rightarrow \dim \Lambda^k V = \binom{n}{k}$.

<u>Pf</u> - By adding distinct new dummy elements, may assume $|A_i| = a$, $|B_i| = b$.

Assume $[n]$ = ground set $= \bigcup_{i \in [m]} (A_i \cup B_i)$

Let $V = \mathbb{R}^{a+b}$ and pick n vectors $v_1, \dots, v_n$ in general position in V.

That is, every set of $\leq (a+b)$ of v_i's is linearly independent.

- For each $i \in [m]$, let

$$y_i = \bigwedge_{j \in A_i} v_j \in \Lambda^a V$$

$$z_i = \bigwedge_{k \in B_i} v_k \in \Lambda^b V$$

- $A_i \cap B_i = \emptyset \Rightarrow y_i \wedge z_i \neq 0 \in \Lambda^{a+b} V$ ①

$\forall i<j$: $A_i \cap B_j \neq \emptyset \Rightarrow y_i \wedge z_j = 0$ ②

- It suffices to prove $y_1, \dots, y_m \in \Lambda^a V$ are linearly indep. $\Longrightarrow m \leq \dim(\Lambda^a V)$

$$= \binom{a+b}{a}$$

Suppose $\alpha_1 y_1 + \dots + \alpha_m y_m = 0$

NTS $\alpha_i = 0 \quad \forall i \in [m]$

$$0 = 0 \wedge z_m = (\alpha_1 y_1 + \dots + \alpha_m y_m) \wedge z_m$$

$$= \alpha_1 y_1 \wedge z_m + \dots + \alpha_m y_m \wedge z_m \overset{②}{=} \alpha_m \underbrace{y_m \wedge z_m}_{① \neq 0}$$

$\Rightarrow \alpha_m = 0 \Rightarrow \alpha_1 y_1 + \cdots + \alpha_{m-1} y_{m-1} = 0$

$\cdots\cdots$ $\boxed{\ddot{\smile}}$

§ Applications of Bollobás set-pairs ineq.

① <u>Exer</u> Prove LYM ineq using Bollobás set-pairs ineq

② Saturation number

Given a k-gr (k-uniform hypergraph) $\mathcal{H}$

we say a k-gr G is $\mathcal{H}$-saturated if

G is $\mathcal{H}$-free but adding any edge to G

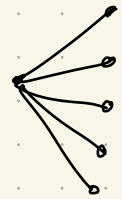
would create a copy of $\mathcal{H}$. (maximal $\mathcal{H}$-free)

Saturation number of $\mathcal{H}$

$$\mathrm{sat}(n, \mathcal{H}) = \min\left\{ e(G) : \begin{array}{l} G \ \ n \text{ vxs} \\ \mathcal{H}\text{-saturated} \end{array} \right\}$$

<u>Example</u> $H = \triangle$ Mantel : $\frac{n^2}{4}$

Max

$\mathrm{sat}(n, \triangle)$  $K_{1,n-1}$

$n-1$ edges

<u>Exer</u> (Bollobás 65) Let $n \geq t \geq k \geq 2$

$$\Rightarrow \quad \mathrm{sat}\left(n, \underline{K_t^{(k)}}\right) = \binom{n}{k} - \binom{n-t+k}{k}$$

t-vertex complete k-gr.