Lecture 11

Last time

Conj (Sidorenko) $\forall$ bip. H $\forall$ G ← host gr

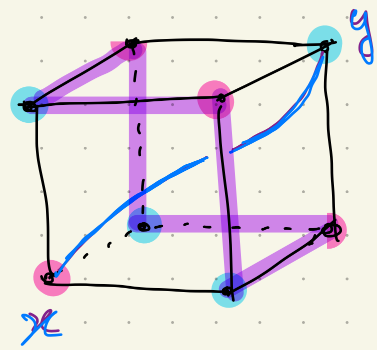
$$\Rightarrow \quad t(H,G) \geq t(K_2,G)^{e(H)}$$

Rewrite $\quad t(H,G) = \dfrac{\hom(H,G)}{n^{|V(H)|}} \geq p^{e(H)}$

$\Leftrightarrow$ Conj: $\forall$ bip H $\forall G$

$$\hom(H,G) \geq n^{|V(H)|} \cdot p^{e(H)}$$

§ 3-dim cube Q_3



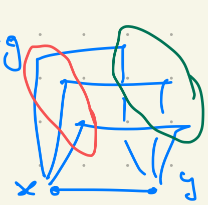
KST $\quad ex(n,Q_3) \leq n^{2-c}$

FAKS $\quad Q_3$ 3-reg $\Rightarrow \leq n^{2-1/3} = n^{5/3}$

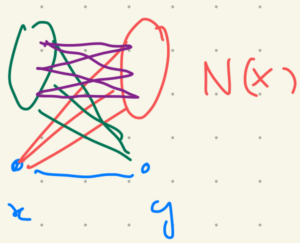
Thm $\quad ex(n, Q_3^+) = O(n^{8/5})$ $\qquad Q_3^+ = Q_3$ + long diagonal

Rmk: For lower bd, we only have $\Omega(n^{3/2})$

Idea $\quad$ • Supersaturation for C_4 $\Rightarrow$ (averaging)



$\exists$ e in many 4-cycles



Find C_6 in $G[N(x),N(y)]$

Pf. NTS $\forall$ n-vx gr G w/ $C \cdot n^{8/5}$ edges $\Rightarrow$ Q_3^+ in G.

- Supersaturation for 4-cycle.

Prop $\forall$ n-vx G w/ large n, $d(G) \geq 2\sqrt{n} \Rightarrow \geq \frac{d(G)^4}{8}$ copies of C_4 in G.

- By regularization trick of E-Sim.

$\Rightarrow$ May assume G is K-regular

$$C n^{3/5} \leq \delta(G) \leq \Delta(G) \leq K\delta(G) \leq K \cdot C \cdot n^{3/5}$$

- Supersaturation $\Rightarrow$ $\# C_4 \geq \frac{d(G)^4}{8}$

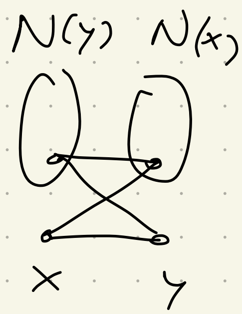
averaging $\Rightarrow$ $\exists$ edge $e = xy$ lying in

$$\geq \frac{\# C_4}{e(G)} \geq \frac{d(G)^4/8}{n \cdot d(G)/2} = \frac{d(G)^3}{4n}$$

$$\geq \frac{C^3}{4} n^{4/5} \text{ many } C_4\text{'s}.$$

$\Rightarrow$ each such C_4 $\Rightarrow$ an edge in $G[N(x), N(y)]$

G' //



$e(G') \geq \frac{C^3}{4} n^{4/5}$

$$|G'| \leq 2\Delta(G) \leq 2K \cdot C \cdot n^{3/5} = N$$

$C \gg K$

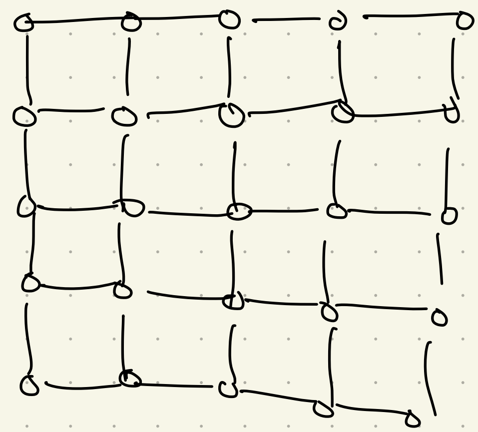
need $\frac{C^3}{4} n^{4/5} > 10 N^{4/3} \Rightarrow C_6$ in G'

Bondy-Sim. $\Rightarrow Q_3^+$ in G.

Another application of supersaturation & hom. ineq.

$t \geq 2$

$G_{t,t}$: $t \times t$ grid



$t=5$

- $C_4 \subseteq G_{t,t}$

$\Rightarrow \quad ex(n, G_{t,t}) \geq ex(n, C_4)$

$$= \Omega(n^{3/2})$$

- Bradač - Janzer - Sudakov - Tomon

$$\Omega\left(t^{1/2} n^{3/2}\right) \leq \text{t×t grid} \leq e^{O(t^5)} n^{3/2}$$

- Gao - Janzer - Liu - Xu

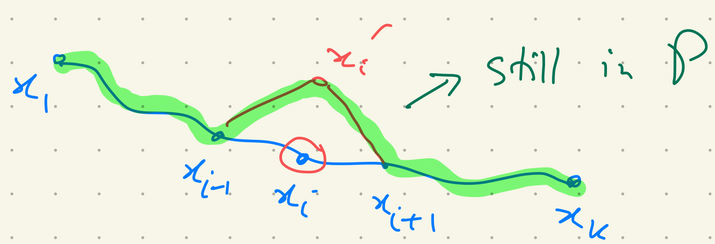
$$ex(n, G_{t,t}) \leq 5t^{3/2} n^{3/2}$$

Would be interesting to determine leading constant.

Idea: Use supersaturation for paths to grow a grid cell by cell.

Def Let $\alpha > 0$, $k \in \mathbb{N}$. A collection $\mathcal{P}$ of labeled paths P_k is α-rich if $\forall$ member $x_1 x_2 \ldots x_k \in \mathcal{P}$ and $\forall\ 2 \leq i \leq k-1$, $\exists^{\geq} \alpha$ distinct vxs x_i' s.t.

$x_1 x_2 \cdots x_{i-1} x_i' x_{i+1} \cdots x_k \in \mathcal{P}$

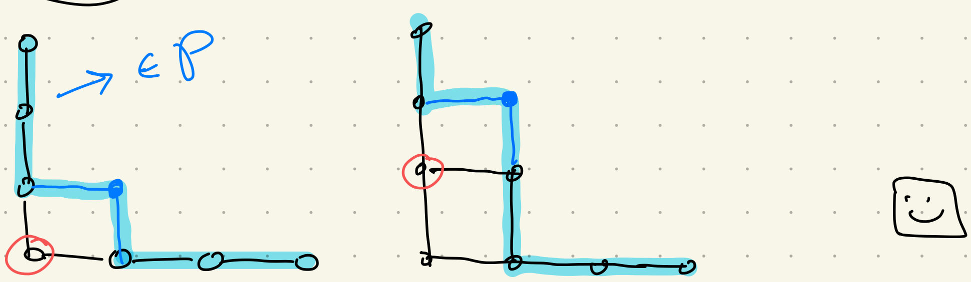


The following prop reduces embedding a grid to finding a collection of rich paths.

Prop Let $\mathcal{P}$ be a non-empty α-rich collection of paths of length $2t-2$ in G.

If $\alpha \geq t^2 \implies$ then G contains a copy of $G_{t,t}$.

Pf: $t=4$



To prove $ex(n, G_{t,t}) \leq 5t^{3/2} n^{3/2}$, just need to show $\forall$ n-vx gr G, $e(G) \geq 5t^{3/2} n^{3/2}$ has an α-rich collection of P_{2t-1}

Lem Let $t \in \mathbb{N}$, n suff. large and G n-vx gr w/ $e(G) \geq 5t^{3/2} n^{3/2} \implies G$ has a non-empty α-rich collection of P_{2t-1} w/ $\alpha = t^2$.

Pf: May assume G is K-almost-regular.

$$d = d(G) \geq 10 t^{3/2} n^{1/2}$$

Let $\mathcal{P}_0$ be the set of all P_{2t-1} in G.

As G is almost regular

$\Rightarrow$ most of hom. of P_{2t-1} to G are injective

So $|\mathcal{P}_0| \geq \frac{1}{2} \hom(P_{2t-1}, G)$

- Deletion process: define a seq of collections of P_{2t-1}

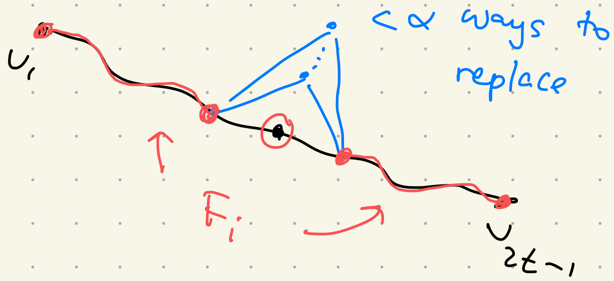
$$\mathcal{P}_0 \supseteq \mathcal{P}_1 \supseteq \mathcal{P}_2 \supseteq \cdots$$

Having defined $\mathcal{P}_i$, we now define $\mathcal{P}_{i+1}$

If $\exists\, P_i \in \mathcal{P}_i$ and a vx $a \in P_i$

s.t. $\#\, P' \in \mathcal{P}_i$ containing $\underbrace{P_i - a}_{F_i}$

$< \alpha$



$\Rightarrow$ delete all paths containing $P - a$ from $\mathcal{P}_i$ to obtain $\mathcal{P}_{i+1}$.

- Let $s = \#$ steps of the whole deletion process.

<u>NTS</u> the final collection $\mathcal{P}_s$ is non-empty.

- As we delete all paths containing F_i at ith step, all the F_i's are distinct.

By Claim below $\Rightarrow$ $\#$ steps $s \leq \#$ different choices for F_i

- Total $\#$ paths deleted in the whole process $\leq s \cdot \alpha \leq \frac{1}{4} \hom(P_{2t-1}, G) \leq \frac{1}{2} |\mathcal{P}_0|$

<u>Claim</u> There are $\leq \frac{1}{4t^2} \hom(P_{2t-1}, G)$ possibilities for F_i.

Lem (Erdős-Sim.) $\forall\, k > \ell$ positive integers s.t. k even $\Rightarrow \forall$ n-vx gr G, we have

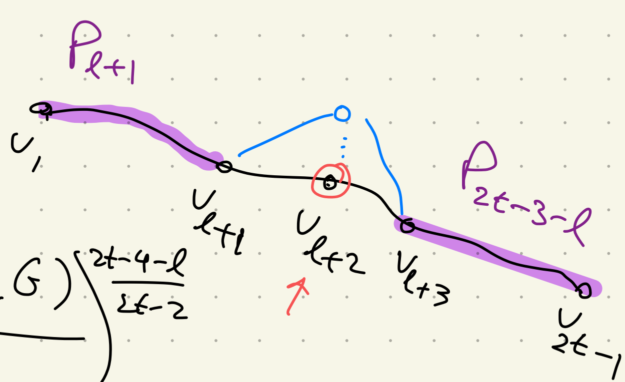
$$\left(\frac{\hom(P_{k+1}, G)}{n}\right)^{1/k} \geq \left(\frac{\hom(P_{\ell+1}, G)}{n}\right)^{1/\ell}$$

<u>Pf of Claim</u> There are $< 2t$ choices for the position of the removed vertex a.

It suffices to show $\forall\ 0 \leq \ell \leq 2t-4$,

$\#$ subgr $\cong P_{\ell+1} \cup P_{2t-3-\ell}$ in $G \leq \frac{1}{8t^3} \hom(P_{2t-1}, G)$

$\leq \hom(P_{\ell+1}, G) \cdot \hom(P_{2t-3-\ell}, G)$



$$\overset{\text{Lem}}{\leq} n\left(\frac{\hom(P_{2t-1}, G)}{n}\right)^{\frac{\ell}{2t-2}} \cdot n\left(\frac{\hom(P_{2t-1}, G)}{n}\right)^{\frac{2t-4-\ell}{2t-2}}$$

$$= n^{\frac{t}{t-1}} \hom(P_{2t-1}, G)^{\frac{t-2}{t-1}}$$

Sidorenko conj for paths $\Rightarrow \hom(P_{2t-1}, G) \geq n \cdot d^{2t-2}$

$\Downarrow$

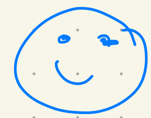
$$\leq \frac{1}{16t^3} \hom(P_{2t-1}, G).$$

§ 3rd pf of Bondy-Sim. using Sidorenko's conj & iterative Cauchy-Schwarz

Independently discovered by
- Janzer – Sudakov
- Kim - Lee - Liu – Tran

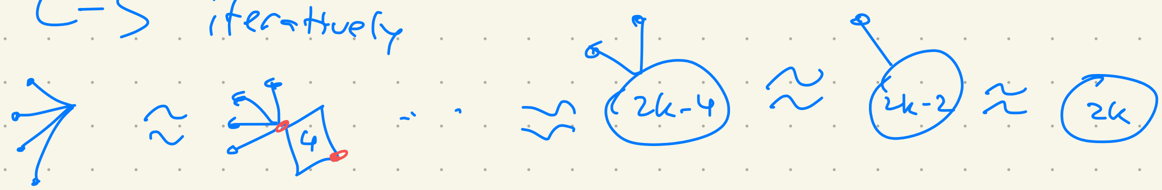
$$ex(n, C_{2k}) = O(n^{1+1/k})$$

$\boxed{G \quad n\text{-vx} \quad K\text{-almost-reg} \quad e(G) \geq C \cdot n^{1+1/k}}$

<u>Idea</u> • If most of C_{2k}-hom are non-degenerate 

Otherwise, we can show that positive fraction of C_{2k}-hom are hom of

• Use C-S iteratively

S_k 

• Final contradiction using Sidorenko

$n^{k+1}p^k \geq_K n\,\Delta(G)^k \geq$  $\geq n^{2k}p^{2k}$

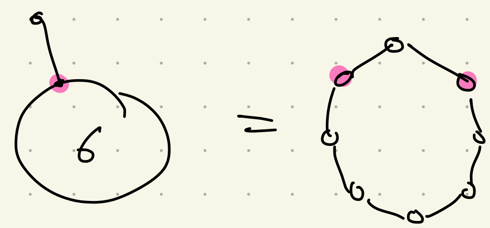
$$\Rightarrow \quad p \leq n^{\frac{1}{k}-1} \quad ↯.$$

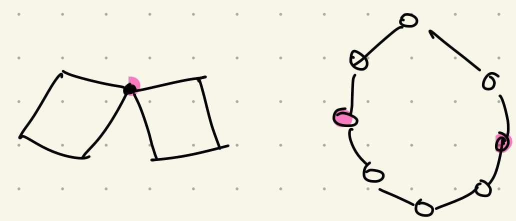
Illustrate it w/ C_8. $\quad ex(n, C_8) \leq O(n^{1+1/4})$

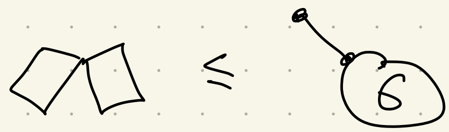
n-vx G $\quad K$-almost-reg

$$d \geq C \cdot n^{1/4} \quad \Longrightarrow \quad C_8 \subseteq G$$
$$p = d/n \approx Cn^{-3/4}$$

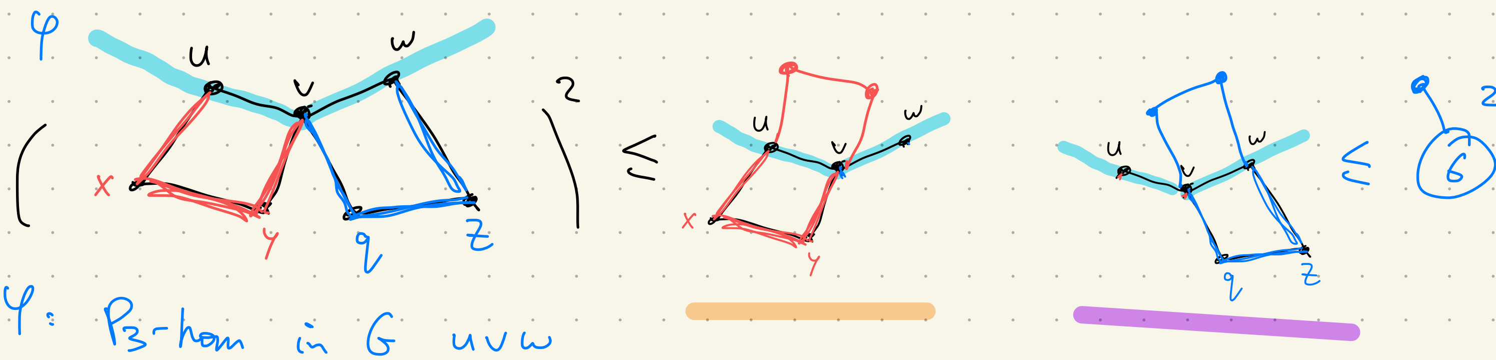
Only two kinds of 'largest' degenerate hom

① 

② 

C-S $\Rightarrow$ 

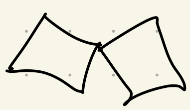
$$\left(\sum a_i b_i\right)^2 \le \left(\sum a_i^2\right)\left(\sum b_i^2\right)$$

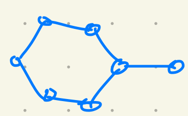


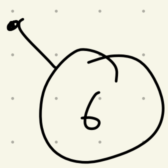
φ: P_3-hom in G uvw

a_φ: # extensions of φ to P_4-hom u x y v

b_φ: # ———————— v q z w

hom( , G$)^2$ $= \left(\sum_\varphi a_\varphi \cdot b_\varphi\right)^2 \le \left(\sum a_\varphi^2\right)\left(\sum b_\varphi^2\right)$

$\le$ hom( , G$)^2$

$\Rightarrow$ hom( , G) $= \Omega\left(\text{hom}(C_{2k}, G)\right)$.