Lecture 10

<u>Recall</u> FAKS: $\forall$ bip. H w/ max-deg $\leq r$ from one side

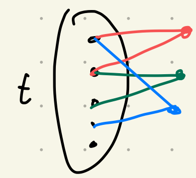
$$\Rightarrow \quad ex(n, H) = O(n^{2-1/r})$$

<u>Exer</u> Use random zooming lem to prove the following.

<u>Def</u> 1-subdivision of K_t

replace each edge of K_t

by internally disjoint paths of length 2.



<u>Prove that</u> $\forall$ n-vx G with cn^2 edges

$\Longrightarrow$ $\exists$ 1-subdivision of K_t in G with

$$t = \Omega(c\sqrt{n})$$

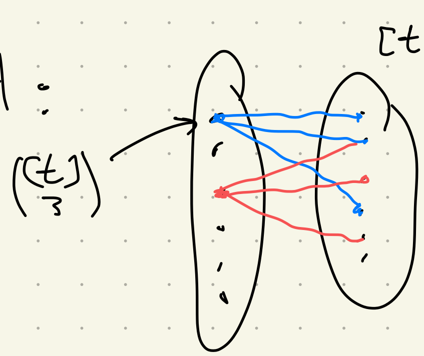
- A recent conj of Conlon–Lee speculates that FAKS bound is tight only when $K_{r,r} \subseteq H$.

<u>Conj</u> (Conlon–Lee) $\forall$ bip. H
- one side deg $\leq r$
- H is $K_{r,r}$-free

$$\Longrightarrow \quad ex(n, H) = O(n^{2 - 1/r - c_H})$$

- Known: True for $r = 2$.
- Open for all $r \geq 3$.

Open H:



$K_{3,3}$-free

3-bdd

$ex(n, H) \leq n^{2 - 1/3 - \epsilon}$?

§ Sidorenko's conj

Sidorenko's conj is a conj about graph homomorphisms ineq relating subgr densities and edge density.

<u>Recall</u> $\varphi : V(H) \to V(G)$ hom. if it preserves adj.

$$\forall\, uv \in E(H) \Rightarrow \varphi(u)\varphi(v) \in E(G)$$

<u>Notation</u> $\mathrm{Hom}(H,G)$ = set of all hom. from H to G

$$\hom(H,G) = |\mathrm{Hom}(H,G)|$$

<u>Def</u> The homomorphism density of H in G is the fraction of maps $V(H) \to V(G)$ that are hom.

i.e.
$$t(H,G) = \frac{\hom(H,G)}{|V(G)|^{|V(H)|}}$$

- $t(H,G)$ is the prob. that a unif. random map is a hom.

<u>Exer</u> $t(H,\cdot)$ is invariant wrt blowup

$$t(H,G) = t(H, G[s])$$

$$p = t(K_2, G) = \text{edge density of } G$$

n vxs

labeled

$$= \frac{\hom(K_2, G)}{n^2} = \frac{2e(G)}{n^2}$$

Example

$$t(K_2, K_2) = \frac{2}{2^2} = \frac{1}{2}$$

$$t(K_3, K_2) = \frac{0}{2^3} = 0$$

Conj (Sidorenko) $\forall$ bip. H $\forall$ G

host gr

$$\Rightarrow \quad t(H, G) \geq t(K_2, G)^{e(H)}$$

Rewrite

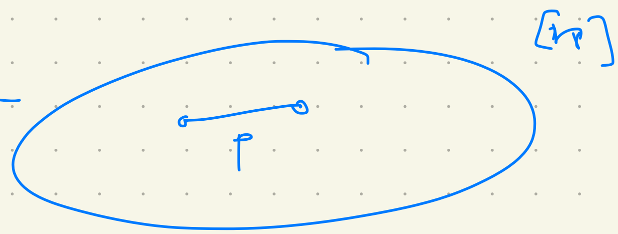
$$t(H, G) = \frac{\hom(H, G)}{n^{|V(H)|}} \geq p^{e(H)}$$

$\Longleftrightarrow$ Conj: $\forall$ bip H $\forall G$

$$\hom(H, G) \geq n^{|V(H)|} \cdot p^{e(H)}$$

The RHS above is the expected # hom. from H to a random gr $G(n, p)$

expected edge density p



So Sidorenko's conj is saying that among all gr with given edge density, random gr minimizes the # H-homomorphisms.

Rmk Necessary to require H to be bip.

$$t(K_3, K_2) \qquad t(K_2, K_2) = 1/2$$

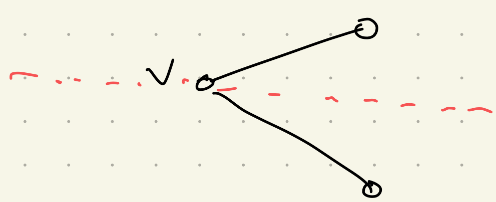
$$\underset{0}{\overset{\shortparallel}{}} \quad \geq t(K_2,K_2)^3 = \frac{1}{8}$$

Known

- Sidorenko: H trees, even cycle, $K_{s,t}$
- Hatami: hypercube
- Conlon-Fox-Sudakov: bip H w/ one vx completely joined to the other side.
- Li-Szegedy
- blowups
- finite reflection gp

$\Big\}$ Conlon-Lee.

Warm up — P_3



NTS $\forall G$, $p = \frac{2e(G)}{n^2}$

$$\hom(P_3, G) \geq n^3 p^2$$

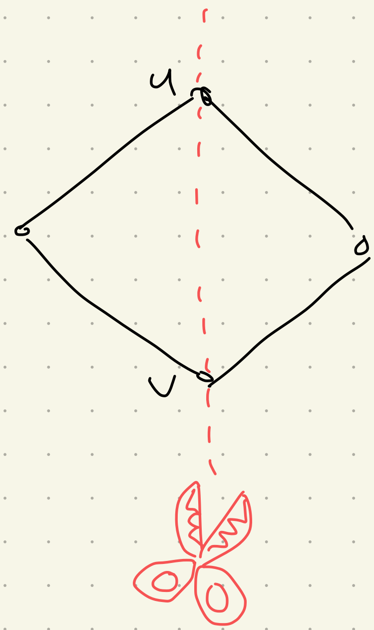
C-S

$$\left(\sum a_i b_i\right)^2 \leq \left(\sum a_i^2\right)\left(\sum b_i^2\right)$$

$$\hom(P_3, G) = \sum_{v \in V(G)} d(v)^2 \overset{C\text{-}S}{\geq} \frac{1}{n}\Big(\underbrace{\sum_{v \in V(G)} d(v)}_{\substack{2e(G) \\ = n^2 p}}\Big)^2$$

$$= \frac{1}{n}\cdot(n^2 p)^2 = n^3 p^2$$

- C_4



<u>NTS</u> $\hom(C_4, G) \geq n^4 p^4$

$$\hom(C_4, G) = \sum_{u,v \in V(G)} \underbrace{d(u,v)^2}_{\text{codegree}}$$

$$\overset{C\text{-}S}{\geq} \frac{1}{n^2}\Big(\sum_{u,v \in V(G)} d(u,v)\Big)^2 = \frac{1}{n^2} \hom(P_3, G)^2$$

$$\geq \frac{1}{n^2}(n^3 p^2)^2 = n^4 p^4$$

- C_{2k} : NTS $\hom(C_{2k}, G) \geq n^{2k} p^{2k}$

Let A be the adjacency matrix of an n-vx gr G.

0,1 symm. matrix

Since A is real & symm.

has real eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$

Prop $\forall G \quad \lambda_1(G) \geq d(G)$ average degree

Prop $\forall G$, A adj matrix of G

$$\operatorname{tr}(A^{2k}) = \sum_{i=1}^{n} \lambda_i^{2k} = \#\text{ closed } 2k\text{-walks in } G$$
$$= \hom(C_{2k}, G)$$

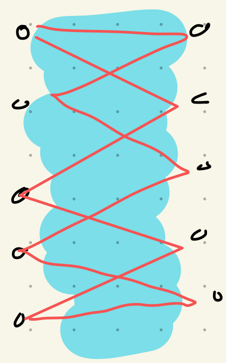
pf $\quad \hom(C_{2k}, G) = \sum_{i=1}^{n} \lambda_i^{2k} \geq \lambda_1^{2k} \geq d(G)^{2k}$

$$d(G) = \frac{2e(G)}{n} = np \longrightarrow = (np)^{2k}$$

Smallest open case (Sidorenko)

Mobius band $K_{5,5} \setminus C_{10}$

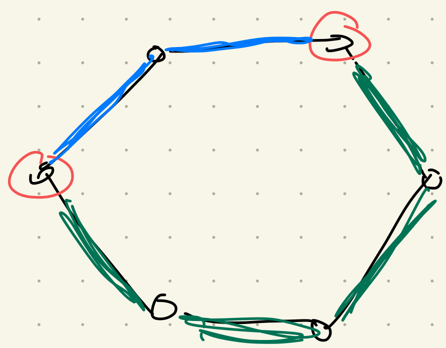
remove edges of C_{10} from $K_{5,5}$

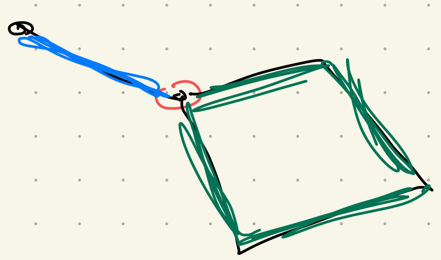


Application of Sidorenko's conj.

$$\mathrm{ex}(n, C_6) = O(n^{1+1/3}) = O(n^{4/3})$$

<u>Def</u> A homomorphism is degenerate if it is not injective





Goal: Many C_6-hom in graph w/ $\gg n^{4/3}$ edges are injective $\Rightarrow$ many copies of C_6

<u>Pf</u> By regularization lem, may assume our gr is K-almost-regular.

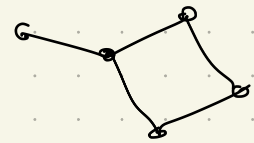
n-vx G: $\Delta(G) \le K\delta(G)$

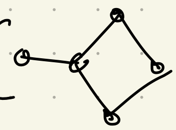
$e(G) \ge C n^{4/3}$, $(C \gg K)$

Write $p = \dfrac{2e(G)}{n^2} = \dfrac{2C}{n^{2/3}}$.

- Sidorenko $\Rightarrow$ $\hom(C_6, G) \ge n^6 p^6$

Suppose all of these C_6-hom are degenerate

Since all degenerate C_6-hom is a hom for 

$\Rightarrow$ hom( , G) = hom(C_6, G) $\geq n^6 p^6$

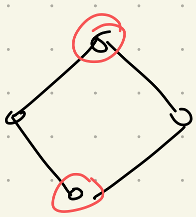
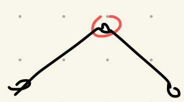
$\geq$ hom(C_4, G) $\cdot \Delta(G)$

$d(G) = np$

C_4-hom count large!

$$\Rightarrow \hom(C_4, G) \geq \frac{n^6 p^6}{Knp} = \frac{1}{K} n^5 p^5$$

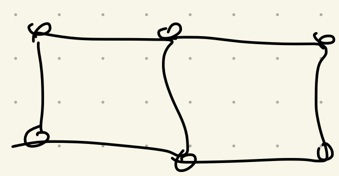
Most of them are copies of C_4

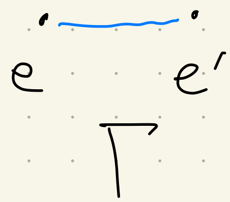
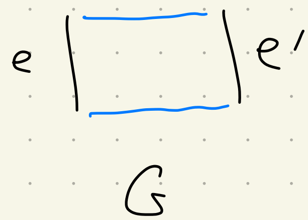
$$\hom(P_3, G) \leq n \cdot \Delta(G)^2 = K^2 n^3 p^2$$

$$\ll \frac{1}{K} n^5 p^5 \leq \hom(C_4, G)$$

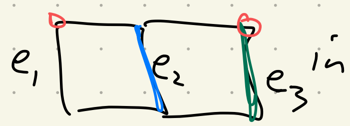
$$\Rightarrow \geq \frac{1}{2K} n^5 p^5 \text{ copies of } C_4$$

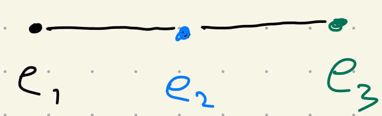
We shall embed  $= C_6$

Build aux. gr Γ : $V(\Gamma) = E(G)$

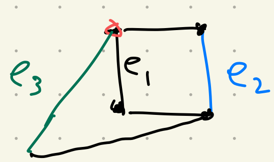

$\Longleftarrow$


$e \sim e'$ in Γ $\Longleftrightarrow$ $e \cup e' \cong C_4$ in G

Then finding  in G

$\Longleftrightarrow$ finding  Γ

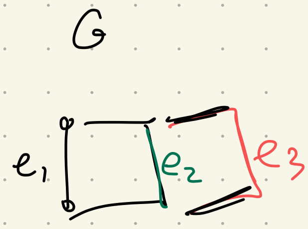
s.t. $V(e_1) \cap V(e_3) = \phi$

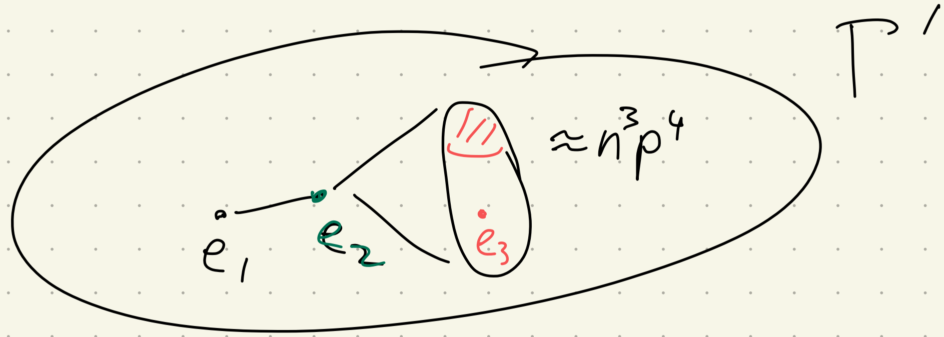
BAD : $V(e_1) \cap V(e_3) \neq \phi$ 

$$d(\Gamma) \text{ average deg} \geq \frac{\# C_4 \text{ in } G}{e(G)} \geq \frac{\frac{1}{2K} n^5 p^5}{n^2 p/2}$$

$$\approx n^3 p^4$$

$\Rightarrow \Gamma' \subseteq \Gamma$ w./ $\delta(\Gamma') \geq \frac{1}{2} d(\Gamma) \approx n^3 p^4$





$$d_{\Gamma'}(e_2) \gtrsim n^3p^4 \gg 2\Delta(G) = Knp$$

↖ # bad choices for e_3

<u>Exer</u> Prove $ex(n, C_{2k}) = O(n^{1+1/k})$

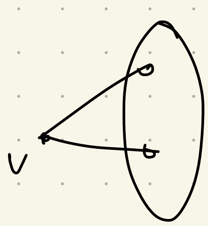
using the above idea.

• Supersaturation for 4-cycle.

<u>Prp</u> $\forall$ n-vx G w/ large n,

$$d(G) \geq 2\sqrt{n} \implies \geq \frac{d(G)^4}{8} \text{ copies of } C_4 \text{ in } G.$$

<u>Pf</u>: • $d = d(G)$



cherries $K_{1,2}$ in $G \geq \sum_{v \in V(G)} \binom{d(v)}{2}$

Jensen ineq

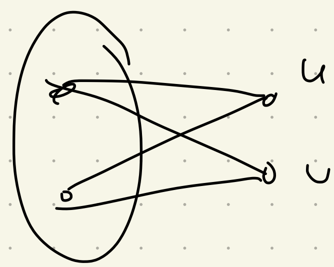
$$\geq n \binom{\frac{1}{n}\sum d(v)}{2} = n\binom{d}{2}$$

- average codegree of a pair of vxs in G

$$D = \frac{1}{\binom{n}{2}} \sum_{(u,v)\in\binom{V(G)}{2}} d(u,v) = \frac{\#\text{cherries}}{\binom{n}{2}} \geq \frac{n\binom{d}{2}}{\binom{n}{2}}$$

$$\geq \frac{d(d-1)}{n-1} \geq 2$$

$$\#C_4 \geq \sum_{u,v} \binom{d(u,v)}{2}$$



$$\overset{\text{Jensen}}{\geq} \binom{n}{2}\binom{D}{2} \geq \frac{d^4}{8}.$$