Lecture 8 Bip. Turán (degenerate)

ESS: $ex(n, H) = \left(1 - \frac{1}{\chi(H)-1} + o(1)\right) \frac{n^2}{2}$

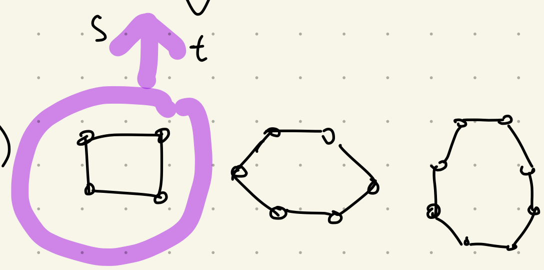
Survey: Füredi & Simonovits

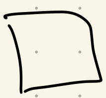
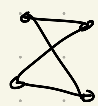
$\chi(H) = 2$ bip. $\Rightarrow$ $o(n^2)$

Actually $\forall$ bip. H $\Rightarrow$ $O(n^{2-c_H})$

Most basic bip. gr:

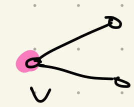
1)  complete bip. gr $K_{s,t}$

2)  . . . $C_{2\ell}$

 $= C_4 = K_{2,2} =$ 

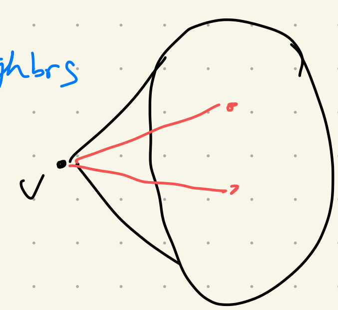
<u>Thm</u> $ex(n, C_4) \leq \frac{n}{4}\left(1 + \sqrt{4n-3}\right) = \left(\frac{1}{2} + o(1)\right) n^{3/2}$

Later: $ex(n, C_4) = \left(\frac{1}{2} + o(1)\right) n^{3/2}$.

<u>Idea</u>: Double count cherries $K_{1,2}$ 

<u>Pf</u> Let G be an n-vx C_4-free gr.

① mid pt: v $\Rightarrow$ ② pick two neighbrs

$\#$ cherries $= \sum_{v \in V(G)} \binom{d(v)}{2}$ 

Jensen $f(x) = \binom{x}{2}$

$\geq n \cdot \binom{\frac{1}{n}\sum d(v)}{2}$

$$= n\binom{\frac{1}{n}2e(G)}{2}$$

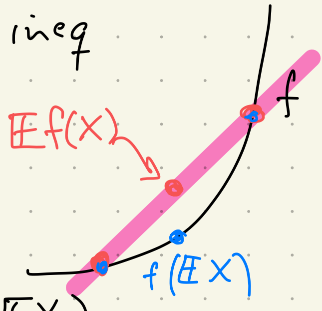
$$= \frac{2e(G)^2}{n} - e(G)$$

Jensen's ineq

f convex

$\forall$ r.v. X

$\Rightarrow \mathbb{E}f(X) \geq f(\mathbb{E}X)$



On the other hand, no two cherries share endpts o.w. $\Rightarrow C_4$ ↯

$\Rightarrow$ # Cherries $\leq \binom{n}{2}$

Exer: $\forall s \leq t \quad ex(n, K_{s,t}) \leq t \cdot n^{2-1/s}$

Thm (Kővári–Sós–Turán) $\forall s \leq t$

$$ex(n, K_{s,t}) \leq \left(\tfrac{1}{2}+o(1)\right)(t-1)^{1/s}\, n^{2-1/s}$$

Major open problem: lower bound

Known $\Omega(n^{2-1/s})$
- $s = 2,3$
- $t \gg s$ (· Kollár–Rónyai–Szabó · Alon–Rónyai–Szabó · Bukh $t \gtrsim 2^s$)

Best $\Omega(n^{5/3}) \leq ex(n, K_{4,4}) \leq O(n^{7/4})$

— C_4-free gr & Sidon set —

Def: A set $S = \{a_1, \dots, a_k\} \subseteq \mathbb{N}$ is a Sidon set if all pairwise sums $a_i + a_j$ are distinct.

In other words, $a+b=c+d$ has only trivial sol$^{\underline{n}}$ $\{a,b\}=\{c,d\}$ in S.

<u>Q</u>: How large can a Sidon set in $[n]$ be?

<u>Ex</u> powers of 2: $1, 2, 2^2, 2^3, \dots, 2^k = n$

size $\log_2 n$

<u>Thm</u> The largest Sidon set in $[n]$ has size $(1+o(1))\sqrt{n}$.

<u>Ex</u>: 1) Prove the weaker upp bd that $\forall$ Sidon set in $[n]$ has size $\leq 2\sqrt{n}$.

2) Use the Thm above to construct n-vx C_4-free gr w/ $\Omega(n^{3/2})$ edges

<u>Thm</u> (Erdős - Rényi - Sós) $ex(n, C_4) \geq (\frac{1}{2} - o(1))\, n^{3/2}$.

<u>Pf</u>: For large enough n, there is a prime number p between $(1-o(1))\sqrt{n+1}$ & $\sqrt{n+1}$.

G: $\cdot\ V(G) = \mathbb{F}_p^2 \setminus \{(0,0)\}$

$\cdot\ E(G) = \{(a,b) \sim (x,y) : ax+by=1\}$

'line' $\sim$ 'pt' $\Leftrightarrow$ 'pt on the line'

Every vertex in G has deg p or $p-1$ (if loop)

$$e(G) \geq \frac{1}{2}(p-1)(p^2-1) = \left(\frac{1}{2} - o(1)\right) n^{3/2}$$

Take two distinct vxs (a_i, b_i), $i \in [2]$

$N((a_i, b_i))$ consists of all vxs (x, y) satisfying $a_i x + b_i y = 1$ line in $\mathbb{F}_p^2$

As $(a_1, b_1) \neq (a_2, b_2)$

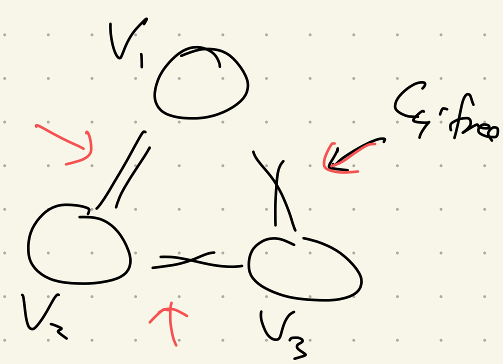
$N((a_1, b_1))$ & $N((a_2, b_2))$ are two different lines which can intersect at ≤ 1 pt.

$\Rightarrow$ (a_1, b_1), (a_2, b_2) have ≤ 1 common neighbr.

Open 3-partite gr G on (n, n, n) vxs

$[V_1, V_2]$, $[V_2, V_3]$, $[V_1, V_3]$

What is max # Δs ?



$$n^{5/3} \leq \# \Delta s \leq n^{7/4}$$

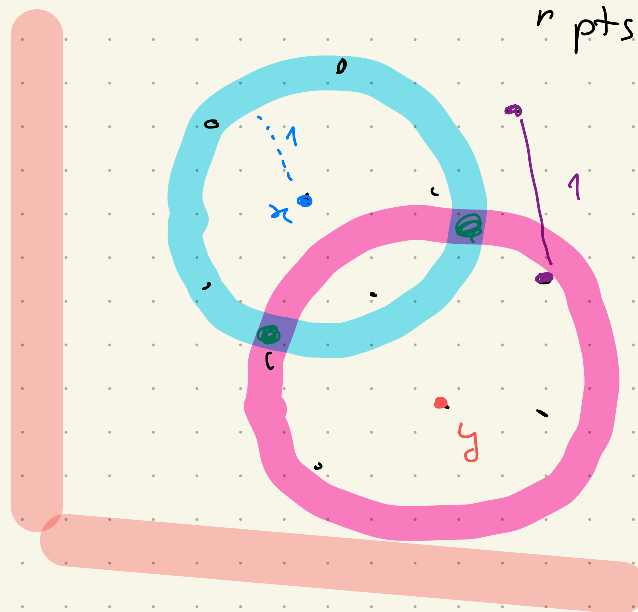
Coulter - Matthews - Timmons

KST : $ex(n, K_{s,t}) = O(n^{2 - 1/s})$, $s \leq t$

Quick application to Erdős unit distance problem.

Prop A set of n pts in $\mathbb{R}^2$ determines

$O(n^{3/2})$ unit distance



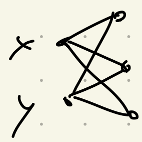
Pf Build an auxiliary gr

G on these n pt, where

two pts are adj if dist( ) = 1

Need to upp bd $e(G)$

Obs G is $K_{2,3}$-free



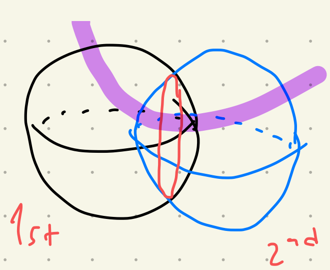
KST 

Rmk best know upp bd is $O(n^{4/3})$

Szemerédi - Trotter.

- Brown's construction: $ex(n, K_{3,3}) = \Omega(n^{5/3})$

  Idea: 3 spheres in $\mathbb{R}^3$ have ≤ 2 pts in common



$V = \mathbb{F}_p^3$

$(a,b,c) \sim (a',b',c')$

$\Leftrightarrow (a-a')^2 + (b-b')^2 + (c-c')^2 = 1$

$N(v)$ corresponds to a sphere around v.

— Even cycles —

<u>Thm</u> (Bondy - Simonovits)

$$ex(n, C_{2k}) = O(n^{1+1/k})$$

Only matching lower bd known: $k = 2, 3, 5$

coming from finite geometry

<u>Open</u> $\Omega(n^{6/5}) \leq ex(n, C_8) \leq O(n^{5/4})$

$\uparrow$

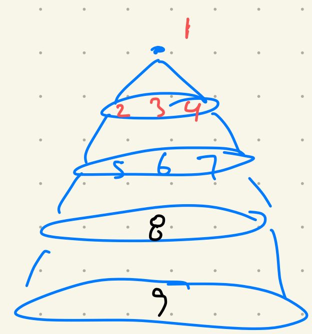
Lazebnik - Ustimenko - Wolder.

<u>Thm</u> (Pikhurko) For large n,

$$ex(n, C_{2k}) \leq 4k \cdot n^{1+1/k}$$

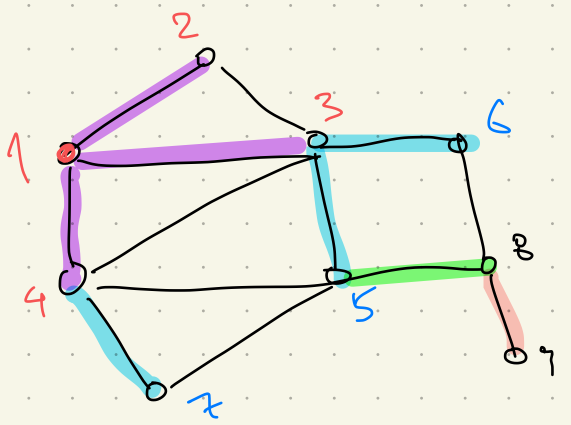
<u>Rmk</u> Best leading constant: $O(\sqrt{k \log k}\, n^{1+1/k})$ [He]

- BFS (Breadth First Search)



Graph Exploration algo. output: spanning tree.

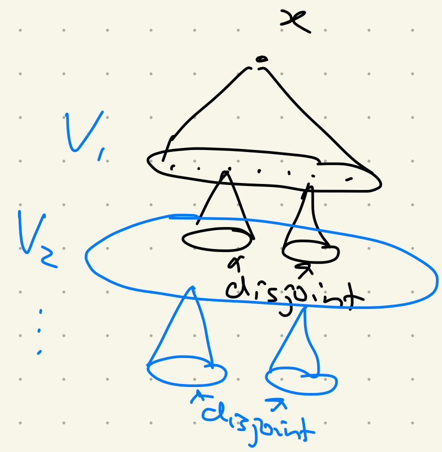
G



Easier: $\mathcal{C}_{2k} = \{C_4, C_6, \dots, C_{2k}\}$

$$ex(n, \mathcal{C}_{2k}) = O(n^{1+1/k})$$

Take a BFS $\Rightarrow$ tree T



$\mathcal{C}_{2k}$-free $\Rightarrow |V_i| \geq \delta(G)^i$

$\forall\, i \leq k$

$$\delta(G)^k \leq |V_k| \leq |V(G)| = n$$

$$\Rightarrow \delta(G) \leq n^{1/k}$$

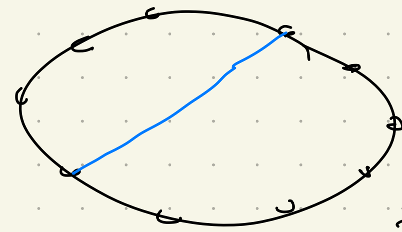
<u>Rmk</u> To forbid only C_{2k}, we need to be more careful. Note that shorter even cycles are (degenerate) homomorphic images of C_{2k}.

The main difficulty in many bip. Turán problem is to control the count of such degenerate homomorphisms.

<u>Prop</u> $\forall\, G$ contains a subgr H w/ $\delta(H) \geq d(G)/2$

<u>Prop</u> $\forall\, G$ contains a bip. subgr H w/ $e(H) \geq e(G)/2$

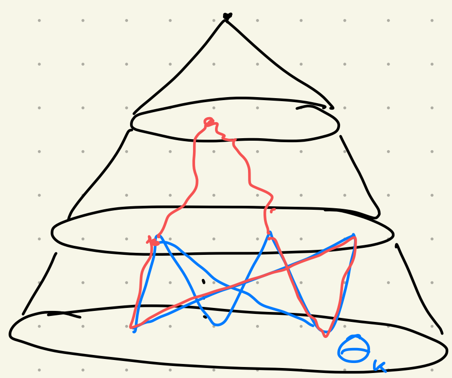
Def A Θ_k-gr is a cycle of length $\geq 2k$ with a chord.



Exer Let $k \geq 3$, H bip. gr w/ $d(H) \geq 2k$

$\Rightarrow$ H contains a Θ_k-gr.

Idea Consider a BFS tree T



- If between any of the first k pairs of consecutive layers, there is a Θ_k-gr $\Rightarrow$ C_{2k}

  By Exer $\Rightarrow$ between layers few edges

  $\Rightarrow$ every vx sends most edges downward

  $\Rightarrow$ layers are expanding again by a factor of $\Omega(d(G))$

  After k steps $\Omega(d^k) = |V_k| \leq n \Rightarrow d \leq n^{1+1/k}$

- To use Θ_k for building C_{2k}, we would need to find many paths of varying length in Θ_k.

Lem Let F be a Θ_k-gr and let $A \cup B = V(F)$ be a non-trivial partition of $V(F)$ $(A, B \neq \emptyset)$

If F is not bipartite w/ bipartition $A \cup B$

$\Longrightarrow$ then $\exists$ A,B-paths of all lengths less than $|F| = n$.