Lecture 7

§ Stability method

Tackle an extremal problem

- Step 1 : Obtain asymp. result ;
- Step 2 : Obtain Stability result;
- Step 3 : Use the structural info. from the Stability to slowly remove imperfection to get an exact result.

Idea

Illustration : Thm $ex(n, C_{2k+1}) = \lfloor \frac{n^2}{4} \rfloor$ $= ex(n, K_3)$

(Recall : special case of color-critical gr)

Step 1 E-S-S : $ex(n, C_{2k+1}) = \frac{n^2}{4} + o(n^2)$

Step 2. E-Sim :

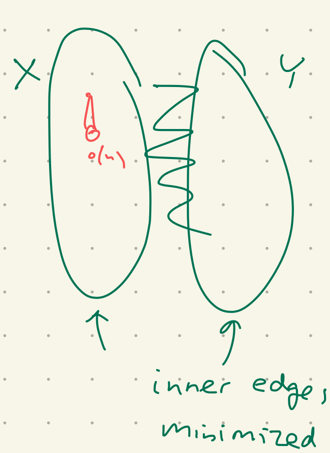
Lem $\forall \alpha > 0 \ \exists n_0$ s.t. TFH $\forall n \geq n_0$

$\forall G$ n-vx $\begin{cases} C_{2k+1}\text{-free} \\ e(G) \geq \frac{n^2}{4} - \alpha n^2 \end{cases} \Longrightarrow$ G can be made bipartite by removing $\leq 3\alpha n^2$ edges.

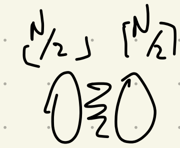
Idea :
- Take max-cut $X \cup Y$

STS $e_G(X) + e_G(Y) = 0$

- $|X| = |Y| \approx n/2$
- (weaker) max inner deg $= o(n)$
- Not a single inner edge allowed



Pf • Let G_{ex} be an N-vx extremal C_{2k+1}-free gr

Lower bd: $e(G_{ex}) \geq \lfloor \frac{N^2}{4} \rfloor =$ 

• boost min deg: find $G \subset G_{ex}$ s.t. $\delta(G) \geq (\frac{1}{2} - \varepsilon)|G|$

Start w/ $G_0 = G_{ex}$

Do: • if $\exists\ v_0 \in V(G_0)$ w. $d_{G_0}(v_0) < (\frac{1}{2} - \varepsilon) N$

Let $G_1 = G_0 - v_0$

At ith step

$\exists\ v_{i-1} \in V(G_{i-1})$ w. $d_{G_{i-1}}(v_{i-1}) < (\frac{1}{2} - \varepsilon)(N-(i-1))$

Let $G_i = G_{i-1} - v_{i-1}$

Claim this process must terminate w/ a desired gr $G_k = G$

Pf

$$e(G_k) \geq \frac{N^2}{4} - \sum_{i=1}^{k} (\tfrac{1}{2} - \varepsilon)(N+1-i)$$

$$\geq \frac{N^2}{4} - (\tfrac{1}{2} - \varepsilon) Nk + (\tfrac{1}{2} - \varepsilon) \sum_{i=1}^{k} (i-1)$$

Write $n = |G_k| = N - k$

$$= \frac{(n+k)^2}{4} - (\tfrac{1}{2} - \varepsilon)(n+k)k + \text{—\,''\,—}$$

$$\cdots = \frac{n^2}{4} + \varepsilon nk + \frac{\varepsilon}{2} k^2 + O(k)$$

When $k = \Omega(N) = \Omega(n) \Rightarrow\ \geq \frac{n^2}{4} + \Omega(n^2)$

$\lightning$ E-S-S.

- We have n-vx $G \subseteq \mathcal{G}_{ex}$ $\begin{cases} C_{2k+1}\text{-free} \\ \delta(G) \geq (\frac{1}{2}-\varepsilon)n \end{cases}$ $\Rightarrow e(G) \geq \frac{n^2}{4} - \frac{\varepsilon}{2}n^2$

- Consider a Max-cut $X \cup Y$ for G

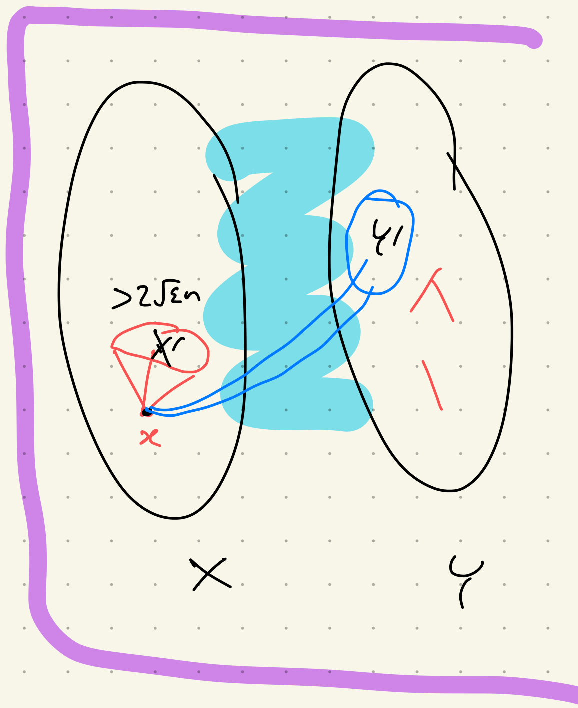
Apply Stability on G

$$\Rightarrow e_G(X) + e_G(Y) \leq \frac{3\varepsilon}{2}n^2$$

<u>Claim</u> $|X| = |Y| = \frac{n}{2} \pm 2\sqrt{\varepsilon}n$

<u>Pf</u> O.w.

$$e(G) \leq |X||Y| + e_G(X) + e_G(Y)$$
$$\leq \frac{n^2}{4} - 4\varepsilon n^2 + \frac{3\varepsilon}{2}n^2$$
$$< \frac{n^2}{4} - \frac{\varepsilon}{2}n^2 \quad ↯.$$



<u>Claim</u> $\Delta(G[X]), \Delta(G[Y]) \leq 2\sqrt{\varepsilon}n$.

<u>Pf</u> - Suppose not, $\exists\, x \in X$

$$d(x, X) > 2\sqrt{\varepsilon}n$$

Max-cut $\Rightarrow$ $d(x, Y) \geq d(x, X) > 2\sqrt{\varepsilon}n$

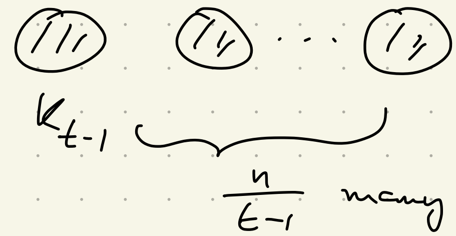
Let $N(x, X) = X'$, $N(x, Y) = Y'$

- As G is C_{2k+1}-free, the bip. gr

$$\Rightarrow G[X', Y'] \text{ is } P_{2k}\text{-free}$$

Thm (Erdős–Gallai) $ex(n, P_t) \le \frac{t-2}{2} \cdot n$

Rmk optimal when $t-1 \mid n$:



$$e(G) \le \underbrace{\frac{3\varepsilon n^2}{2}}_{e_G(X)+e_G(Y)} + \underbrace{|X||Y|}_{\text{cross edges}} - (\underbrace{|X'||Y'| - kn}_{\#\text{ missing edges between } X' \,\&\, Y'})$$

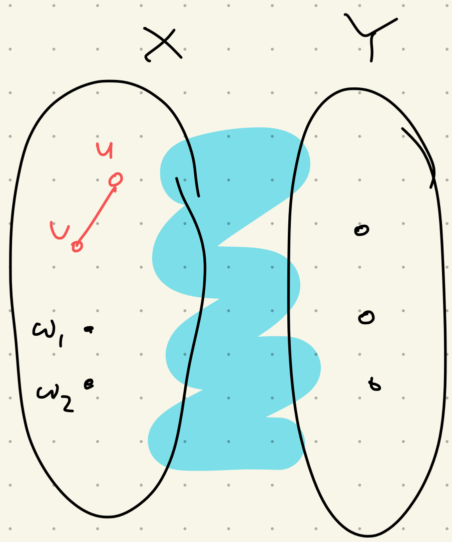
$$\le \frac{3\varepsilon n^2}{2} + \frac{n^2}{4} + kn - 4\varepsilon n^2$$

$$< \frac{n^2}{4} - \frac{\varepsilon}{2} n^2 \qquad \lightning .$$

G
- $|X| = |Y| = \frac{n}{2} \pm 2\sqrt{\varepsilon} n$
- $\delta(G) \geq (\frac{1}{2} - \varepsilon) n$
- $\Delta(G[X]), \Delta(G[Y]) \le 2\sqrt{\varepsilon} n$

We now show no edge in X or Y.

Suppose u, v in X form an edge



$k=3$

C_7-free

$\forall$ vertex $x \in X$, almost full deg to Y:

$$d(x, Y) \geq \delta(G) - \Delta(G[X])$$

$$\geq (\tfrac{1}{2} - \varepsilon) n - 2\sqrt{\varepsilon} n$$

$$\geq (1 - 10\sqrt{\varepsilon}) |Y|$$

Pick arbitrary $w_1, \dots, w_{k-1} \in X - u - v$

$\Rightarrow$ $u, v, w_1, \ldots, w_{k-1}$ have $\geq \left(1-(k+1)10\sqrt{\varepsilon}\right)|Y| \geq 3$.

$\Rightarrow$ $C_{2k+1} \subseteq G$ $\lightning$.

So G is bipartite w/ partition $X \cup Y$.

$$e(G_{ex}) \leq e(G) + \sum_{i=n+1}^{N} \left(\tfrac{1}{2}-\varepsilon\right) i$$

$$< \frac{N^2}{4} \quad \text{if} \quad n < N$$

$\Rightarrow$ G_{ex} must be G $\Rightarrow$ G_{ex} bip

$\Rightarrow$ $G_{ex} \cong T_{N,2}$. ☺

HW sent to

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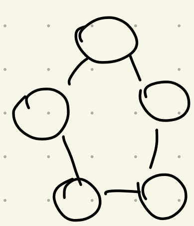
- $\forall$ n-vx Δ-free G

  if $\delta(G) > \frac{2n}{5}$ $\Rightarrow$ G bipartite

Optimal



$n/5$

$\delta(G) = \frac{2n}{5}$

non-bip

Δ-free

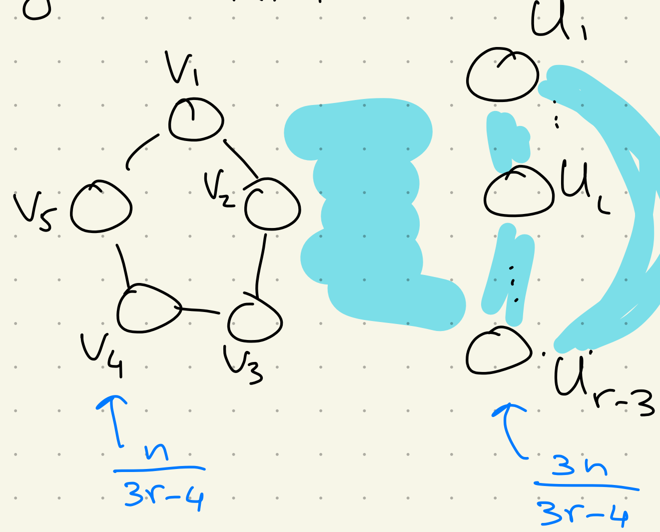
Thm Andrásfai-Erdős-Sós

$\forall$ n-vx K_r-free gr G

$$\delta(G) > \frac{3r-7}{3r-4} \cdot n \implies G \text{ is } (r-1)\text{-partite}.$$

We give a short pf by Brandt.

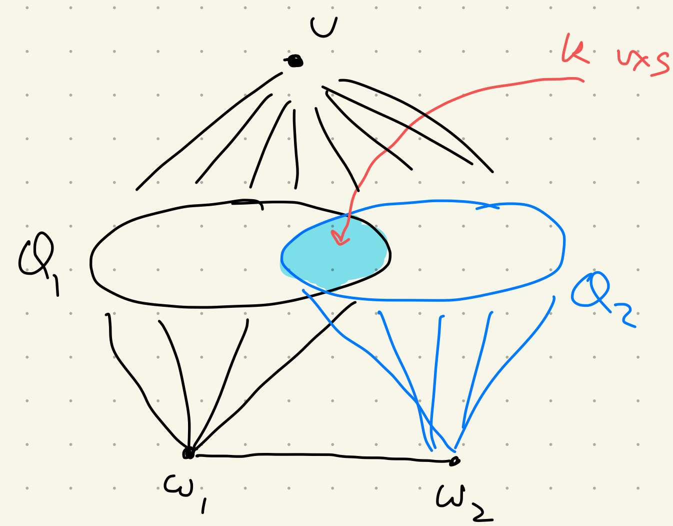
Rmk Bound is tight:



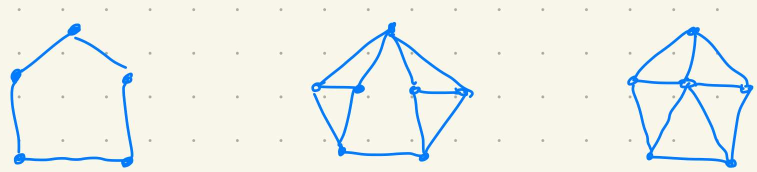
Def (5-wheel-like graph $W_{r,k}$)

$0 \le k \le r-3$

- top vx v
- bottom edge w_1, w_2
- two $(r-2)$-cliques Q_1, Q_2 with $|Q_1 \cap Q_2| = k$
- $v \sim Q_1$ & Q_2
- $w_1 \sim Q_1$
- $w_2 \sim Q_2$



Ex $W_{3,0}$ $W_{4,0}$ $W_{4,1}$



The nice idea of Brandt is the follow fact generalizing the fact that

Exer "every non-bipartite maximal Δ-free gr contains a 5-cycle"

Lem $\forall$ G
- maximal K_r-free
- non-$(r-1)$-partite

$\Longrightarrow$ G contains a 5-wheel-like gr.

Pf: • If G is complete multipartite

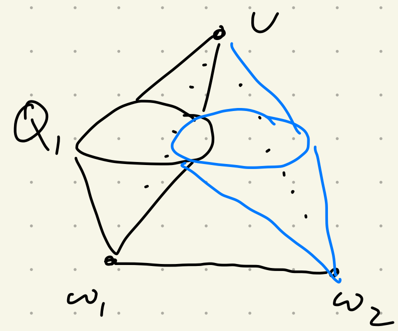
$\Rightarrow$ # parts $\leq r-1$ as G is K_r-free $\lightning$

Thus G is not complete multipartite

$\Rightarrow$ non-adjacency is NOT an equivalence relation

$\Rightarrow$ G contains

$vw_1 \notin E$
$vw_2 \notin E$
$w_1w_2 \in E$



• G is maximal K_r-free $\quad \forall i \in [2]$

$\Rightarrow$ $N(v) \cap N(w_i)$ contains a copy Q_i of K_{r-2}

for otherwise $G + vw_i$ is still K_r-free $\lightning$ maximality of G.

Pf (AES: $\delta(G) > \frac{3r-7}{3r-4} n$, maximal K_r-free $\Rightarrow$ $(r-1)$-partite)

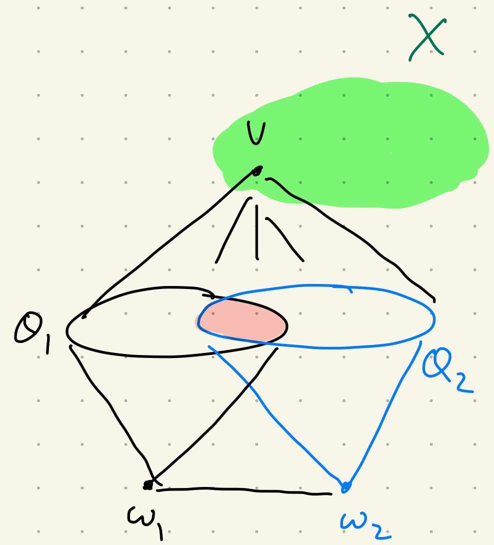
- By adding edges if necessary, we may assume G is maximal K_r-free.

$\xRightarrow{\text{Lem}}$ G contains $W_{r,k}$

- Pick a copy W of $W_{r,k}$ with maximum value k.

Define X = common neighbr of $Q_1 \cap Q_2$

We will bound $|X|$ in two ways.



1) By double counting deg sum in W.

- $\left\{ \begin{array}{l} G \text{ is } K_r\text{-free} \\ W \text{ contains a } K_{r-1} \end{array} \right. \Rightarrow$ $\forall$ vx v v is not completely joined to W

Write $p = |W|$

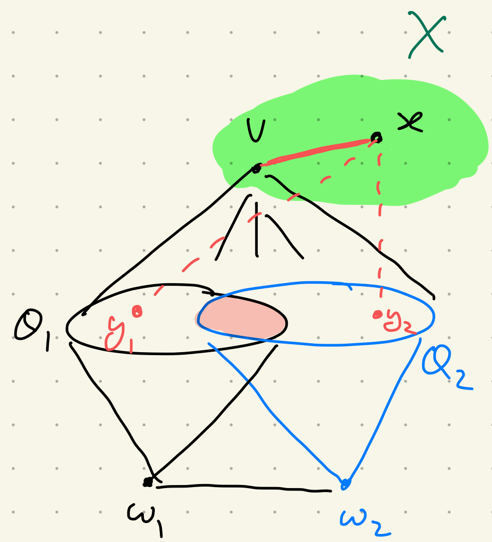
$\Rightarrow$ $d(v, W) \le p-1$

Claim $\forall x \in X$, $d(x, W) \le p-3$

Pf: G is K_r-free

$\Rightarrow$ x has a non-neighbor in $Q_i \cup v$ and $Q_i \cup \{w_i\}$ $\forall$ $i \in [2]$

Case 1 $x \sim v$



By the above obs.

$\Rightarrow$ x has a non-neighbr $y_1 \in Q_1 \setminus Q_2$, $y_2 \in Q_2 \setminus Q_1$

Consider v, w_1, w_2, Q_1', Q_2'

where $Q_i' = Q_i - y_i + x$

$\Rightarrow$ we get $W_{r,k+1}$ ↯ maximality of W.

$\Rightarrow$ $x \not\sim v$ $\Rightarrow$ $d(x, W) \le p-3$ ☺

& x has non-neighbr in (disjoint) $Q_i \cup w_i \setminus (Q_1 \cap Q_2)$

• deg sum in W

$$① \ldots p \cdot \delta(G) \le \sum_{w \in W} d(w) \le \underbrace{(p-3)|X| + (p-1)(n-|X|)}_{\text{deg sum of vxs to } W}$$

• On the other hand, by pigeonhole

as X = common neighbr of $Q_1 \cap Q_2$

$$\Rightarrow \quad |X| \ge k\,\delta(G) - (k-1)n \quad \ldots ②$$

$$① + ② \Rightarrow \delta(G) \le \frac{2r+k-4}{2r+k-1} n \overset{k \le r-3}{<} \frac{3r-7}{3r-4} n. \ ↯ ☺$$

AES: n-vx Δ-free, $\delta(G) > \frac{2n}{5} \Rightarrow$ bip

$\chi(G) \leq 2$

Q: How much can we lower the min-deg if we only require bdd chr. # ?

This the Chromatic threshold problem.

Extensions

Def $ex(n, T, H)$ = max # copies of Γ in an n-vx H-free gr

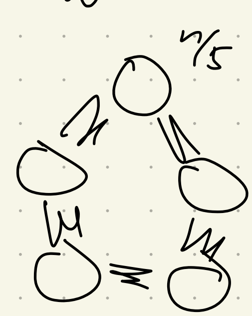
$$ex(n, H) = ex(n, K_2, H)$$

- Erdős - Zykov solve the clique case:

$\forall\ s<t$ $\quad ex(n, K_s, K_t)$ is maximized by $T_{n,t-1}$

- Alon - Shikhelman 16
- Gishboliner - Shapira (cycles)
- Grzesik, Hatami - Hladky - Král' - Norin - Razborov

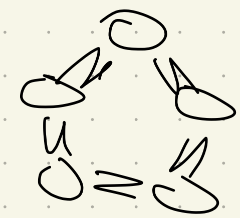
$$ex(n, C_5, K_3) \leq \left(\tfrac{n}{5}\right)^5$$



Two conj of Erdős ($250)

<u>Conj</u> (Sparse halves) $\forall$ n-vx Δ-free gr G contains a set of $n/2$ vxs inducing $\leq n^2/50$ edges.

<u>Rmk</u> Best possible if true:



<u>Conj</u> $\forall$ n-vx Δ-free gr can be made bipartite by removing $\leq n^2/25$ edges