Lecture 6

Previously: - Supersaturation via local sampling / averaging

- Application: $\pi(H) = \pi(H[t])$ ← blowups

$\longrightarrow$ ESS

above $\pi(\pi)$

- Moon-Moser $\longrightarrow$ Supersaturation

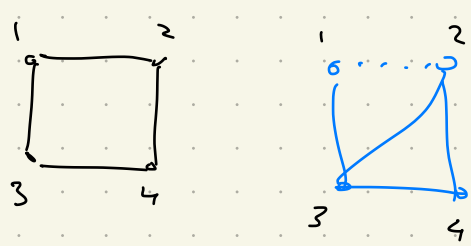
Today: Below the Turán density: Stability

When an extremal problem, such as Turán's thm, has a unique extremal configuration, we often see stability. That is, almost extremal config. are structurally close to the unique extremal config.

<u>Def</u> (Edit-distance) Given two n-vertex graphs G and H, the edit-distance

$$\Delta(G,H) = \min_{\sigma: V(G)\to V(H)} |E(G) \triangle E(\sigma(G))|$$

$(E(G)\setminus E(H)) \cup (E(H)\setminus E(G))$: $|E(G) \triangle E(H)| = 2$



<u>Thm</u> (Erdős - Simonovits) $\forall \alpha > 0$ and $\forall H$ w/ $\chi(H) = r+1$, $\exists\, n_0 = n_0(\alpha, H)$ s.t. T.F.H. for all $n \geq n_0$.

Let G be an n-vx gr.

if $\begin{cases} H\text{-free} \\ e(G) \geq \left(1 - \frac{1}{r}\right)\frac{n^2}{2} - \alpha n^2 \end{cases}$

$\Longrightarrow$ then G can be made r-partite by removing at most $3\alpha n^2$ edges.

In particular, $\Delta(G, T_{n,r}) \leq 7\sqrt{\alpha}\, n^2$

<u>Exer</u> • Prove "In particular" part:

Let $K = K_{n_1, \dots, n_r}$ be an n-vx complete r-part. gr w/ $e(K) \geq e(T_{n,r}) - 2t$

Show that $\sum_{i=1}^{r} \left(n_i - n/r\right)^2 \leq 4t$ and

$$\Delta(K, T_{n,r}) \leq 2n \cdot \sqrt{t/r}$$

This exer also shows that if a complete r-partite gr has size $\approx T_{n,r}$, then its partite size $\approx n/r$.

- Perfect Stability for Turán problem

<u>Thm</u> (Füredi) Let G be an n-vx K_{r+1}-free gr with $e(G) = e(T_{n,r}) - t$

$\Longrightarrow$ Then, G can be made r-partite by removing $\leq t$ edges.

(i.e. $\exists\, G' \subseteq G$ w/ $e(G') \geq e(G) - t$ & $\chi(G') \leq r$)

Furthermore, $\exists$ a complete r-partite gr K w/ $V(K) = V(G)$ s.t. $\Delta(G, K) \leq 3t$

$\hookrightarrow$ Exer.

<u>Rmk</u>: 1) Perfect Stability does not have $\alpha > 0$, n_0, it works for all n and t

When $t = 0 \Longrightarrow$ Turán's thm.

2) It gives a linear relation between the edit-distance $\Delta(G, K)$ and $e(T_{n,r}) - e(G)$.

<u>Idea</u> Erdős's degree majorization argument

2nd pf Mantel / Turán's thm

Thm (Füredi) Let G be an n-vx K_{r+1}-free gr with $e(G) = e(T_{n,r}) - t$

$\Rightarrow$ Then, G can be made r-partite by removing $\leq t$ edges.

Pf (Perfect Stability)

It suffices to find a partition

$V(G) = V_1 \cup \cdots \cup V_r$ s.t.

inner edges $\sum_{i=1}^{r} e_G(V_i) \leq t$

- Pick a vertex x_1 of maximum degree

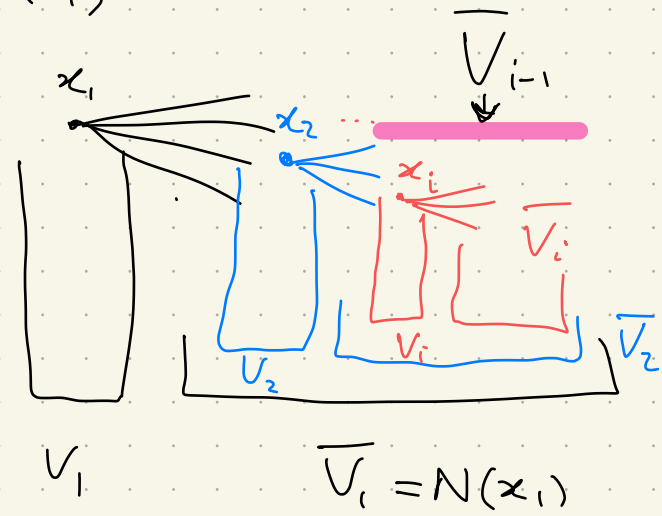
Let $V_1 = V \setminus N(x_1)$, $\overline{V}_1 = N(x_1)$

$\forall x \in V_1$, $d(x) \leq d(x_1)$

- Double count deg sum of V_1

$$|V_1||\overline{V}_1| \geq \sum_{x \in V_1} d(x)$$

$$= e(V_1, \overline{V}_1) + 2e_G(V_1)$$



Zoom into $\overline{V}_1$ & repeat this process.

In general, let $\overline{V}_0 = V(G)$

For every $i \geq 1$ do the following

- Pick x_i: max deg vx in $G[\overline{V}_{i-1}]$
- Let $V_i = \overline{V}_{i-1} \setminus N(x_i)$

$\overline{V}_i = \overline{V}_{i-1} \cap N(x_i)$

By choice of $x_i \Rightarrow \forall x \in V_i$, $d(x, \overline{V}_{i-1}) \leq |\overline{V}_i|$

$$\Rightarrow |V_i||\bar{V}_i| \geq \sum_{x \in V_i} d(x, \bar{V}_{i-1}) = \underbrace{e(V_i, \bar{V}_i)}_{\text{cross edges}} + \underbrace{2e_G(V_i)}_{\text{inner edges}}$$

(♡) ...

At the end of the process

$$\Rightarrow V_1 \cup \cdots \cup V_s.$$

Note $s \leq r$ as $x_1, \dots, x_r$ form a clique & G is K_{r+1}-free.

Summing (♡)

$$\Rightarrow e(T_{n,r}) \geq e(K_{|V_1|, \dots, |V_s|})$$

inner edges

$$= \sum_{i=1}^{s} |V_i||\bar{V}_i| \geq e(G) + \underbrace{\sum_{i=1}^{s} e_G(V_i)}$$

$$\Rightarrow \sum_{i=1}^{s} e_G(V_i) \leq e(T_{n,r}) - e(G) = t.$$

Some recent work on Stability

- degree majorization useful for hyp. [Mubayi 2006 "A hypgr extension of Turán's thm"]
- Liu - Pikhurko - Sharifzadeh - Staden (Abstract framework ...)
- Balogh - Clemen - Lauru - Lidický - Pfender ("Make K_{r+1}-free gr r-partite")

· Liu - Mubayi - Reiher "A unified approach to hypgr Stability"

$\mathcal{G}$: K_{r+1}-free gr

λ : $e(G)$

[Perfect Stability]

Let $\mathcal{G}$ be a gr family and λ be a parameter defined on members of $\mathcal{G}$.

Say λ is perfectly stable if $\exists\, C > 0$ s.t.

$\forall\, G \in \mathcal{G}$ of order n, $\exists$ a complete partite gr K of order n s.t. $\Delta(G, K) \leq C(\lambda(n) - \lambda(G))$

where $\lambda(n) = \max\{\lambda(G) : G \in \mathcal{G},\ |G| = n\}$.

Possible direction of research: perfect stability in other setting / discrete structure.

Füredi: Perfect Stability $\Rightarrow$ E-Sim Stability

Lem (Ruzsa - Szemerédi removal lem)

$\forall\, \alpha > 0$ $\forall\, H$ w/ $\chi(H) = t$, $\exists\, n_0 = n_0(\alpha, H)$ s.t. T.F.H. for all $n \geq n_0$

$\forall\, G$ n-vx H-free $\Rightarrow$ $\exists\, G' \subseteq G$ $\begin{cases} e(G') \geq e(G) - \alpha n^2 \\ G' \text{ is } H\text{-hom-free} \end{cases}$

In particular, G' is K_t-free

Pf • Removal lem

$\Rightarrow \exists G_1 \subseteq G$

$$\begin{cases} \cdot\ e(G_1) \geq e(G) - \alpha n^2 \\ \qquad\qquad \geq e(T_{n,r}) - 2\alpha n^2 \\ \cdot\ G_1 \text{ is } K_{r+1}\text{-free} \end{cases}$$

Thm (Erdős - Simonovits) $\forall \alpha > 0$ and $\forall H$ w/ $\chi(H) = r+1$, $\exists n_0 = n_0(\alpha, H)$ s.t. T.F.H. for all $n \geq n_0$.

Let G be an n-vx gr.

if $\begin{cases} H\text{-free} \\ e(G) \geq (1 - \frac{1}{r})\frac{n^2}{2} - \alpha n^2 \end{cases}$

$\Rightarrow$ then G can be made r-partite by removing at most $3\alpha n^2$ edges.

• Perfect Stability w/ $t = 2\alpha n^2$

$\Rightarrow \exists\ G_2 \subseteq G_1$ s.t. $\begin{cases} \cdot\ e(G_2) \geq e(G_1) - t \\ \qquad\qquad \geq e(G) - 3\alpha n^2 \\ \cdot\ G_2 \text{ is } r\text{-partite.} \end{cases}$

One way of using stability result is to boost an asymptotic sol$^{\underline{n}}$ to an exact one.

§ Stability method

Tackle an extremal problem

- Step 1 : Obtain asymp. result ;
- Step 2 : Obtain Stability result ;
- Step 3 : Use the structural info. from the Stability to slowly remove imperfection to get an exact result.

Idea

Illustration: <u>Thm</u> $ex(n, C_{2k+1}) = \lfloor \frac{n^2}{4} \rfloor$

$= ex(n, K_3)$

(Recall : special case of color-critical gr)

Step 1 E-S-S : $ex(n, C_{2k+1}) = \frac{n^2}{4} + o(n^2)$

Step 2. E-Sim :

<u>Lem</u> $\forall \alpha > 0 \ \exists n_0$ s.t. TFH $\forall n \geq n_0$

$\forall G$ n-vx $\begin{cases} C_{2k+1}\text{-free} \\ e(G) \geq \frac{n^2}{4} - \alpha n^2 \end{cases} \Longrightarrow$ G can be made bipartite by removing $\leq 3\alpha n^2$ edges.

<u>Idea</u> :

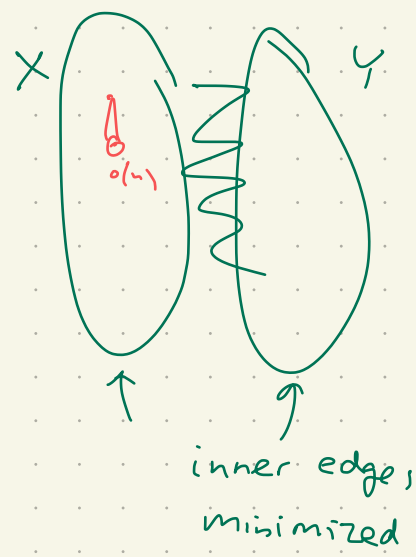
- Take max-cut $X \cup Y$

  <u>STS</u> $e_G(X) + e_G(Y) = 0$

- $|X| = |Y| \approx n/2$

- (weaker) max inner deg $= o(n)$

- Not a single inner edge allowed



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