Lecture 5

Last time defined Turán density of H

$$\pi(H) = \lim_{n\to\infty} \frac{ex(n,H)}{\binom{n}{2}}$$

- To prove upp bd on $\pi(H) \leq \alpha$

  NTS $\forall \varepsilon > 0 \; \exists n_0(\varepsilon)$ s.t. $\forall n \geq n_0$

  $$ex(n,H) \leq (\alpha+\varepsilon)\binom{n}{2}$$

- lower bd $\pi(H) \geq \alpha$

  NTS $\{G_n\}_{n\to\infty}$ H-free $e(G) \geq (\alpha-\varepsilon)\binom{n}{2}$.

- $\pi(H)$ exists for all H via local averaging

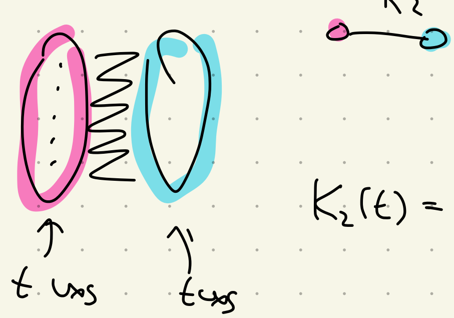
ESS : $\pi(H) = 1 - \dfrac{1}{\chi(H)-1}$

H bip : $\pi(H) = 0$

$\pi(K_2(t)) = 0$

blowup of a graph

$H(t)$ blowup of H



$K_2(t) = K_{t,t}$

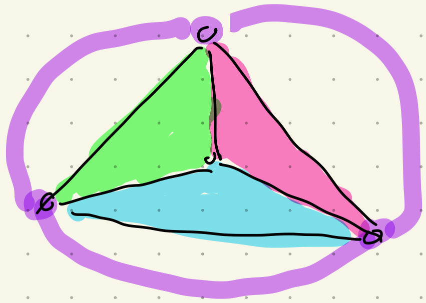
Def $r \in \mathbb{N}$, an r-uniform hypergraph $\mathcal{H}$ on vertex set $[n]$ is a family of r-subsets of $[n]$.

hyperedge

r-unif hyp. $\overset{\text{call it}}{\Longleftrightarrow}$ r-graph

Ex 2-graph = graph

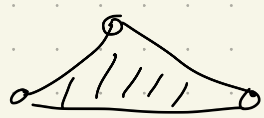
Ex complete 3-unif hyp. on 4 vxs, tetrahedron



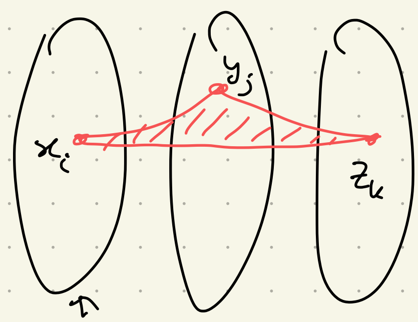
$K_{t,t}$ complete ~~2~~ r-partite ~~2~~ r-graph

$= K_2(t)$ blowup of a single ~~2~~ r-edge

Ex $r=3$ $\quad K^{(3)}_{t,t,t}$



single 3-edge



n vxs in each

$x_i y_j z_k$ edge $\forall i,j,k \in [n]$

Exer $\quad \pi(\mathcal{H}) = \lim_{n\to\infty} \dfrac{ex(n,\mathcal{H})}{\binom{n}{r}}$ exists for all $\mathcal{H}$.

r-graph

Thm (Erdős 64) $\forall t \geq 1$, $\forall r \geq 2$

$$\pi\left(K_r^{(r)}(t)\right) = 0$$

Supersaturation phenomenon: edge density above $\pi(H)$

$$e(G) > (\pi(H)+\varepsilon)\binom{n}{2} \Longrightarrow \text{Many copies of } H$$

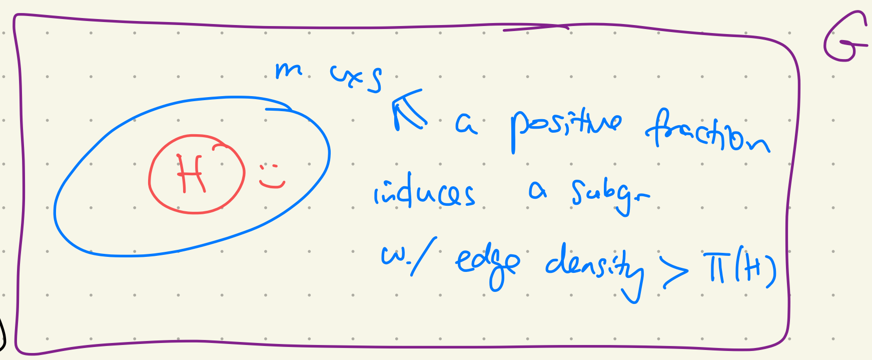
positive fraction

Thm $\forall \varepsilon > 0$, $\forall H$ on h vxs
$\exists \delta > 0$ & $n_0 > 0$ s.t. TFH for all $n \geq n_0$.

$\forall$ n-vx gr G w/ $e(G) \geq (\pi(H)+\varepsilon)\binom{n}{2}$

$\Rightarrow$ G contains $\geq \delta \binom{n}{h}$ copies of H

Idea local averaging



Pf: By defn of $\pi(H)$, we can take a sufficiently large m (depending only on ε & H) s.t.

$$ex(m, H) \leq \left(\pi(H) + \frac{\varepsilon}{4}\right)\binom{m}{2}$$

Claim $\geq \frac{\varepsilon}{4}\binom{n}{m}$ m-sets in $V(G)$ inducing a subgr w/ $> \left(\pi(H)+\frac{\varepsilon}{4}\right)\binom{m}{2}$ edges

Pf Suppose o.w.,

$$\Rightarrow \sum_{M \in \binom{V(G)}{m}} e(G[M]) \leq \frac{\varepsilon}{4}\binom{n}{m}\cdot\binom{m}{2} + \binom{n}{m}\left(\pi(H)+\frac{\varepsilon}{4}\right)\binom{m}{2}$$

$$= \left(\pi(H)+\frac{\varepsilon}{2}\right)\binom{n}{m}\binom{m}{2}$$

$$= \binom{n-2}{m-2}\cdot e(G) \geq (\pi(H)+\varepsilon)\binom{n}{2}\binom{n-2}{m-2}$$

$\Rightarrow\Leftarrow$:)

Each one of these m-set contains a copy of H

each copy of H is contained in

$$\leq \binom{n-h}{m-h} \text{ m-sets}$$

$$\Rightarrow \# \text{ copies of } H \text{ in } G \geq \frac{\frac{\varepsilon}{4}\binom{n}{m}}{\binom{n-h}{m-h}} \geq \underbrace{\boxed{\frac{\varepsilon}{4\binom{m}{h}}}}_{=\,\delta} \binom{n}{h}$$

Application Supersaturation

$\overset{(*)}{\Rightarrow}$ any gr & its blowup have the same Turán density

$\Rightarrow$ Er-St-Sim.

We prove $^{(*)}$ only for cliques.

Prop $\forall t \geq 1, \forall r \geq 2 \quad \pi(K_r) = \pi(K_r(t))$

Pf $(\leq)$ trivial as $K_r \subseteq K_r(t)$

$(\geq)$ $\underset{\text{NTS}}{\forall \varepsilon\ \exists n_0\ \forall n \geq n_0}$ $ex(n, K_r(t)) \leq (\pi(K_r)+\varepsilon)\binom{n}{2}$

i.e. $\forall e(G) \geq (\pi(K_r)+\varepsilon)\binom{n}{2}$ need to embed $K_r(t) \subseteq G$

By supersaturation, G has $\delta\binom{n}{r}$ copies of K_r

Build auxiliary r-unif hyp $\mathcal{F}$

$V(\mathcal{F}) = V(G)$, $E(\mathcal{F})$: hyperedges are copies of K_r in G

$$\Rightarrow \quad e(\mathcal{F}) \geq \delta \binom{n}{r}$$

As $\pi(K_r^{(r)}(t)) = 0 \Rightarrow K_r^{(r)}(t) \subseteq \mathcal{F}$

which corresponds to a copy of $K_r(t)$ in G ☺

Pf3 of ESS via Supersaturation

Fix H w/ $\chi(H) = r+1$ NTS $\pi(H) = 1 - \frac{1}{r}$

Suffices to show $\pi(H) = \pi(K_{r+1})$ Turán's thm

$(\leq)$

Note : $H \subseteq K_{r+1}(t)$, $t = |H|$

$$\Rightarrow \pi(H) \leq \pi(K_{r+1}(t)) \overset{\text{Prop}}{=} \pi(K_{r+1})$$

$(\geq)$ $\pi(H) \geq \pi(K_{r+1})$ consider : $T_{n,r}$ H-free ☺

- Multicolor ver.

A graph is (K_3, K_3)-free if $E(G)$ can be 2-colored with no monochromatic Δs.

Exer 1 $\quad ex(n, (K_3, K_3)) = \max\ e(G) :$ $|G| = n$
G is (K_3, K_3)-free

Determine $ex(n, (K_3, K_3))$.

Exer 2 $\forall$ G n-vx w/ $ex(n, K_r) + 1$ edges

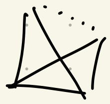
$\Rightarrow$ G has a copy of K_r

Prove that in fact G contains a copy of K_{r+1}^-

(K_{r+1} w./ one edge removed)

Rmk $ex(n, K_{r+1}^-) = ex(n, K_r)$

Ex $r = 3$



H is color-critical if $\exists$ edge $e \in H$

s.t. removing e from H reduces its chromatic number

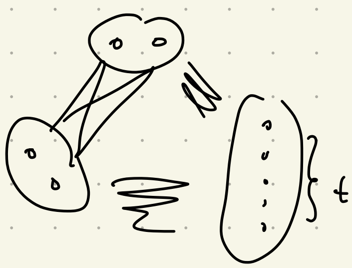
Ex K_{r+1}^-, odd cycle

Thm $\forall$ color-critical H, $ex(n, H) = ex(n, K_{\chi(H)})$

Rmk error term for general H

Thm (Bollobás - Erdős - Simonovits - Szemerédi)

$$ex(n, K_{2,2,t}) = \frac{n^2}{4} + 2 \cdot \underbrace{ex\left(\frac{n}{2}, K_{2,2}\right)}_{\text{extr. \# of}}$$



Error term is controlled by a family of bip. graphs related to H (decomposition family)

§ Moon-Moser ineq

relating clique densities in a gr.

Notation: write $k_r(G) = \#\, K_r$ in G

<u>Thm</u> (Moon-Moser) $\forall$ n-vx gr G $\forall r \geq 2$

$$\Rightarrow \quad \frac{k_{r+1}(G)}{k_r(G)} \geq \frac{1}{r^2-1}\left(r^2 \frac{k_r(G)}{k_{r-1}(G)} - n \right)$$

We can use M-M to bound clique density from below using edge-density. $\forall x > r-1$, $\binom{x}{r} := \frac{x(x-1)\cdots(x-r+1)}{r!}$

$= 0$ if $x \leq r-1$.

<u>Cor</u> If $e(G) = \left(1 - \frac{1}{x}\right)\frac{n^2}{2}$ for some $x \in \mathbb{R}^+$

$$\Rightarrow \text{ then } \quad k_r(G) \geq \binom{x}{r}\left(\frac{n}{x}\right)^r$$

<u>Exer</u> Pf 2 of Supersaturation via M-M. Derive Cor from M-M.

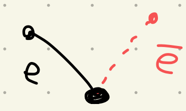
We prove the $r=2$ case of M-M & leave general case as exer.

<u>Pf</u>: For $r=2$, NTS

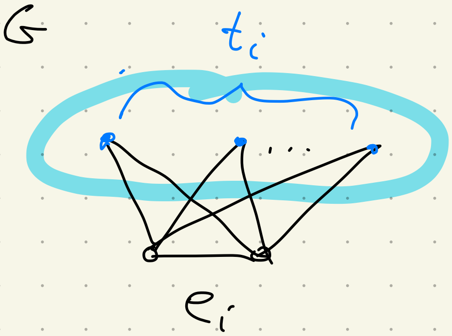
$$\frac{k_3(G)}{e(G)} \geq \frac{1}{3}\left(\frac{4\, e(G)}{n} - n\right)$$

Idea : double counting # pairs $(e, \bar{e}) = P$

where $e \in E(G)$
$\bar{e} \notin E(G)$
$|e \cap \bar{e}| = 1$



- Let $e_1, \dots, e_m$ be all edges of G
  $v_1, \dots, v_n$ be all vxs of G
- Let $t_i = \# \Delta s$ containing e_i
  $d_i = d(v_i)$



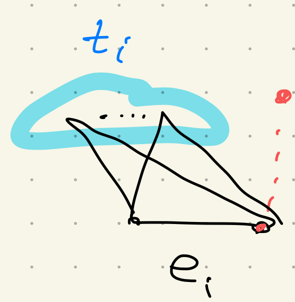
On one hand,

$$(1) \quad P = \sum_{i=1}^{n} d_i(n-1-d_i) \overset{\text{Jensen ineq}}{\leq} n \cdot \bar{d}(n-1-\bar{d})$$

average deg $\quad \bar{d} = \dfrac{\sum_{i=1}^{n} d_i}{n} = \dfrac{2m}{n}$

$$= 2(n-1)m - \frac{(2m)^2}{n}$$

On the other hand,



$$(2) \quad P \geq \sum_{i=1}^{m} (n-2-t_i)$$

$$= m(n-2) - \sum_{i=1}^{m} t_i \equiv 3k_3(G)$$

$$= m(n-2) - 3k_3(G)$$

(1) & (2) $\Rightarrow$ ☺

- Clique density vs blowup

Recall 1st pf of ESS showed

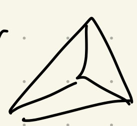
$$e(G) \geq \left(1 - \frac{1}{r-1} + \varepsilon\right)\binom{n}{2} \Rightarrow \exists K_r(t)$$

$$t = \Omega(\log n)$$

- $\log n$ is optimal by considering random graphs.

$$\begin{tikzcd}
 & \#K_r \text{ in } G \text{ is } \Omega(n^r) \arrow[dr, "\text{Nikiforov}"] & \\
e(G) \geq \left(1 - \frac{1}{r-1} + \varepsilon\right)\binom{n}{2} \arrow[ur, "\text{supersaturation}"] \arrow[rr, "\text{ESS}"'] & & \exists K_r(t),\ t = \Omega(\log n)
\end{tikzcd}$$

Rmk Determining Turán density of hypergraphs is much harder than the graph problem

$\pi($  $)$ unknown major open problem.

Even though we do not know the values of Turán densities of most of the hypergr, we can still prove that supersaturation phenomenon occurs for all hypgr.

Exer State the supersaturation result for r-graph & prove it.

Next time Stability (below Turán density)