The uniform Turán density

Def: Given $d \in [0,1]$ and $\eta > 0$, a 3-graph $H = (V, E)$ is called uniformly (d,η)-dense if for every $U \subseteq V$, we have $|U^{(3)} \cap E| \geq d\binom{|U|}{3} - \eta |V|^3$.

Def: The uniform Turán density of a 3-graph F is

$$\pi_{\therefore}(F) = \sup\{d \in [0,1] : \text{for every } \eta > 0 \text{ and } n \in \mathbb{N}, \text{ there is an } F\text{-free, uniformly } (d,\eta)\text{-dense 3-graph } H \text{ with } |V(H)| \geq n\}$$

Pbm (Erdős, Sós): Determine $\pi_{\therefore}(K_4^{(3)-})$ and $\pi_{\therefore}(K_4)$

Thm: (Glebov, Král', Volec and Reiher, Rödl, Schacht)

$$\pi_{\therefore}(K_4^{(3)-}) = \frac{1}{4}$$

Constr: Consider a random tournament T_n on $[n]$. That means for each $\{i,j\} \in [n]$, one of the two orientations (i,j) and (j,i) is chosen independently at random with probability $1/2$. Then let $H(T_n)$ be the hypergraph on the same vertex set for which the edges are given by cyclically oriented triangles of T_n, i.e. $\{i,j,k\}$ forms an edge if $(i,j),(j,k)$, and (k,i) appear in T_n or $(j,i),(i,k),(k,j)$.

Azuma-Hoeffding yields that for every $\eta > 0$, $H(T_n)$ is asymptotically almost surely uniformly $(\frac{1}{4}, \eta)$-dense (as $n \to \infty$). In addition, it can easily be seen that $H(T_n)$ is $K_4^{(3)-}$-free. This implies $\pi_{\therefore}(K_4^{(3)-}) \geq \frac{1}{4}$.

<u>Def:</u> Let $\mathcal{J}$ be a set of indices, for all $ij \in \mathcal{J}^{(2)}$, let $\mathcal{P}^{ij}$ be a set of vertices which are disjoint for $ij \neq i'j'$, and for $ijk \in \mathcal{J}^{(3)}$, let $\mathcal{A}^{ijk}$ be a 3-partite 3-graph with partition classes $\mathcal{P}^{ij}$, $\mathcal{P}^{ik}$, and $\mathcal{P}^{jk}$. Then the 3-graph $\mathcal{A}$ with vertex set $V(\mathcal{A}) = \bigcup_{ij \in \mathcal{J}^{(2)}} \mathcal{P}^{ij}$ and edge set $E(\mathcal{A}) = \bigcup_{ijk \in \mathcal{J}^{(3)}} E(\mathcal{A}^{ijk})$ is called a reduced hypergraph with index set $\mathcal{J}$, vertex classes $\mathcal{P}^{ij}$ and constituents $\mathcal{A}^{ijk}$. We will say 'let $\mathcal{A} = (\mathcal{J}, \mathcal{P}^{ij}, \mathcal{A}^{ijk})$ be a reduced hypergraph.

<u>Def:</u> Let $\mathcal{A} = (\mathcal{J}, \mathcal{P}^{ij}, \mathcal{A}^{ijk})$ be a reduced hypergraph and let $d \in [0,1]$ be a real number. If $e(\mathcal{A}^{ijk}) \geq d\, |\mathcal{P}^{ij}|\, |\mathcal{P}^{ik}|\, |\mathcal{P}^{jk}|$ holds for all $ijk \in \mathcal{J}^{(3)}$, we say that $\mathcal{A}$ is $(d, \therefore)$-dense.

Def: A reduced map from a 3-graph F to a reduced hypergraph $\mathcal{A} = (\mathcal{I}, \mathcal{P}^{ij}, \mathcal{A}^{ijk})$ is a pair (λ, φ) such that

(1) $\lambda : V(F) \to \mathcal{I}$ and $\varphi : \partial F \to V(\mathcal{A})$

(2) for $uv \in \partial F$, we have $\lambda(u) \neq \lambda(v)$ and $\varphi(uv) \in \mathcal{P}^{\lambda(u)\lambda(v)}$

(3) if $uvw \in E(F)$, then $\varphi(uv)\varphi(uw)\varphi(vw) \in E(\mathcal{A}^{\lambda(u)\lambda(v)\lambda(w)})$.

If there is a reduced map from F to $\mathcal{A}$, we say that $\mathcal{A}$ contains a reduced image of F, otherwise $\mathcal{A}$ is called F-free.

thm (Reiher, Rödl, Schacht) For a 3-graph F, we have

$\pi_{\therefore}(F) = \sup\{d \in [0,1] :$ For every $M \in \mathbb{N}$, there is a $(d, \therefore)$-dense, F-free reduced hypergraph $\mathcal{A}$ with index set $[M]\}$

Proof of $\pi_{\therefore}(K_4^{(3)-}) = \frac{1}{4}$ using the theorem above (Reiher, Rödl, Schacht):

Due to the above theorem and the above construction, it is enough to show that for every $\varepsilon > 0$, there is some large $M \in \mathbb{N}$ such that every $(\frac{1}{4}+\varepsilon, \therefore)$-dense reduced hypergraph with index set $[M]$ contains a reduced image of $K_4^{(3)-}$.

Let thus $\mathcal{A} = ([M], \mathcal{P}^{ij}, \mathcal{A}^{ijk})$ be a $(\frac{1}{4}+\varepsilon, \therefore)$-dense reduced hypergraph. We need to find distinct $i_1, \dots, i_4 \in [M]$ and vertices $p^{i_\alpha i_\beta} \in \mathcal{P}^{i_\alpha i_\beta}$ with $\alpha\beta \in [4]^{(2)}$ such that

$$p^{i_1 i_2} p^{i_1 i_3} p^{i_2 i_3}, \quad p^{i_1 i_2} p^{i_1 i_4} p^{i_2 i_4}, \quad p^{i_1 i_3} p^{i_1 i_4} p^{i_3 i_4} \in E(\mathcal{A}).$$

Let $ijk \in [M]^{(3)}$ with $i<j<k$. Define the bipartite graph Q^i_{jk} on the vertex set $\mathcal{P}^{ij} \cup \mathcal{P}^{ik}$ by placing an edge between $w \in \mathcal{P}^{ij}$ and $v \in \mathcal{P}^{ik}$ if there is a vertex $u \in \mathcal{P}^{jk}$ such that $wvu \in E(\mathcal{A}^{ijk})$.

Similarly, define the bipartite graph Q^i_{j} on the vertex set $\mathcal{P}^{ik} \cup \mathcal{P}^{jk}$ by placing an edge between $v \in \mathcal{P}^{ik}$ and $w \in \mathcal{P}^{jk}$ if there is a vertex $u \in \mathcal{P}^{ij}$ such that $uvw \in E(\mathcal{A}^{ijk})$. Note that by double counting and Cauchy-Schwarz, we have

$$\left(\tfrac{1}{4}+\varepsilon\right) |\mathcal{P}^{ij}| \, |\mathcal{P}^{ik}| \, |\mathcal{P}^{jk}| \le e(\mathcal{A}^{ijk}) \le \sum_{v \in \mathcal{P}^{ik}} d_{Q^i_{jk}}(v) \, d_{Q^k_{ij}}(v)$$

$$\le \sqrt{\sum_{v \in \mathcal{P}^{ik}} d^2_{Q^i_{jk}}(v) \sum_{v \in \mathcal{P}^{ik}} d^2_{Q^k_{ij}}(v)}$$

and so (1) $\displaystyle\sum_{v \in \mathcal{P}^{ik}} d^2_{Q^i_{jk}}(v) \ge \left(\tfrac{1}{4}+\varepsilon\right) |\mathcal{P}^{ij}|^2 \, |\mathcal{P}^{ik}|$ <u>or</u>

(2) $\displaystyle\sum_{v \in \mathcal{P}^{ik}} d^2_{Q^k_{ij}}(v) \ge \left(\tfrac{1}{4}+\varepsilon\right) |\mathcal{P}^{ik}| \, |\mathcal{P}^{jk}|^2$.

By Ramsey, we get M_1 indices, w.l.o.g. $[M_1]$, such that for all $ijk \in [M_1]^{(3)}$ with $i<j<k$, the same option, say (1), holds.
Now we view $\overline{Q^i} = \bigcup_{\substack{jk \in [M_1] \\ i<j<k}} Q^i_{jk}$ as an M_1-partite graph. Observing that a triangle in $\overline{Q^i}$ will yield the desired reduced image of $K_4^{(3)-}$, we finish the proof with the following lemma.

Lemma: For every $\varepsilon > 0$, there is some $m \in \mathbb{N}$ such that if $\mathcal{G}$ is an m-partite graph with non-empty vertex classes $V_1, \dots, V_m$ satisfying

$$\sum_{v \in V_i} d_j(x)^2 \geq \left(\tfrac{1}{4} + \varepsilon\right) |V_i| |V_j|^2 \quad \text{for all } 1 \leq i < j \leq m,$$

(where $d_j(x)$ denotes the degree of x in V_j), then $\mathcal{G}$ contains a triangle.

Proof: Let $m^{-1} \ll m_1^{-1} \ll \delta \ll \varepsilon$ and let $\mathcal{G}$ be as above. Assume $\mathcal{G}$ does not contain a triangle. For distinct $i, j \in [m]$ with $i < j$ and $r \in \mathbb{N}$, set

$$R_{ij}(r) = \{x \in V_i : d_j(x) \geq (\tfrac{1}{2} + r\delta)|V_j|\}.$$

Note that $(\frac{1}{4} + \varepsilon)|V_i||V_j|^2 \leq (\frac{1}{2} + \delta)^2 |V_i \setminus R_{ij}(1)|\, |V_j|^2 + |R_{ij}(1)||V_j|^2$ and so by the above hierarchy, we get $|R_{ij}(1)| \geq \delta |V_i|$. Note further that $|R_{ij}(r)|$ is decreasing in r and is zero for $r > \frac{1}{2\delta}$. So we can set $r(i,j)$ to be the maximum r such that $|R_{ij}(r)| \geq \delta |V_i|$.

By Ramsey, we get that there are m_1 indices, w.l.o.g. $[m_1]$, such that for some r_*, we have $r(i,j) = r_*$ for all $ij \in [m_1]^{(2)}$ with $i < j$. By pigeonhole, there are two indices, say 2 and 3, such that $R_{12}(r_*) \cap R_{13}(r_*) \neq \emptyset$, say $x \in R_{12}(r_*) \cap R_{13}(r_*)$.

Setting $A_2 = N_2(x)$, $B_2 = V_2 \setminus A_2$, $A_3 = N_3(x)$, and $B_3 = A_3 \setminus N_3(x)$, we have for $y \in A_2$ that $d_3(y) \leq |B_3| \leq (\frac{1}{2} - r_* \delta)|V_3|$ and for $y \in B_2 \setminus R_{23}(r_* + 1)$ that $d_3(y) < (\frac{1}{2} + (r_* + 1)\delta)|V_3|$.

Meanwhile, we also have $|R_{23}(r_* + 1)| < \delta |V_2|$.

This gives

$$\left(\frac{1}{4}+\varepsilon\right)|V_2||V_3|^2 \leq \sum_{x\in V_2} d_3(x)^2 \leq |A_2|\left(\frac{1}{2}-r_*\delta\right)^2|V_3| + |B_2|\left(\frac{1}{2}+(r_*+1)\delta\right)^2|V_3|^2 + \delta|V_2||$$

Keeping in mind that $|A_2| \geq \left(\frac{1}{2}+r_*\delta\right)|V_2|$ and $|A_2|+|B_2|=|V_2$
we conclude

$$\left(\frac{1}{4}+\varepsilon\right) \leq \left(\frac{1}{2}-r_*\delta\right)^2\left(\frac{1}{2}+r_*\delta\right) + \left(\frac{1}{2}-r_*\delta\right)\left(\frac{1}{2}+(r_*+1)\delta\right)^2 + \delta$$

contradicting $\delta << \varepsilon$.