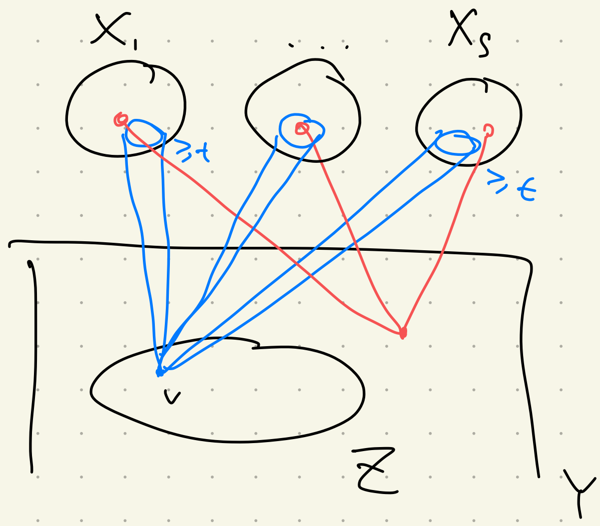
Lecture 4

CONTINUE Pf of E-S-S

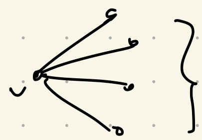
We have G n vxs w/ $\delta(G) \geq (1-\frac{1}{r}+\varepsilon)n$



Claim $Z \subseteq Y$ set of all vxs w/ $\geq t = \frac{r\varepsilon}{4}|X_i|$ neighbrs in each X_i, $i \in [s]$

we have $|Z| \geq \frac{r\varepsilon}{4} n$

Pf Double counting # stars



center $v \in Y$, with exactly one leaf in each X_i

N = # such star

(lower) Fix an s-tuple $(x_1, \dots, x_s) \in X_1 \times X_2 \times \dots \times X_s$

common neighbors of x_i in Y

$$\geq \sum_{i=1}^{s} d(x_i) - (s-1)n - s \cdot \underline{T} \leftarrow \frac{4}{r\varepsilon} \cdot t \quad s \leq r$$

$$\geq (1-\frac{1}{r}+\varepsilon)n \cdot s - (s-1)n - s \cdot T$$

......

$$\geq r\varepsilon n/2$$

(1)…. $N \geq \prod_{i=1}^{s} |X_i| \cdot r\varepsilon n/2$

- $Z' \subseteq Y$ set of vs involved in

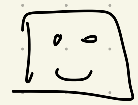
  $\geq \frac{r\varepsilon}{4} \prod_{i=1}^{s} |X_i|$ many such s-stars

Obs : $Z' \subseteq Z$

Pf By defn of Z', $\forall\ z \in Z'$ has $\geq \frac{r\varepsilon}{4}|X_i| = t$ neighbors in each X_i. $\blacksquare$

$$(2) \cdots\ N \leq |Z'| \cdot \prod_{i=1}^{s} |X_i| + n \cdot \frac{r\varepsilon}{4} \prod_{i=1}^{s} |X_i|$$

$$(1) \,\&\, (2) \Rightarrow |Z| \geq |Z'| \geq \frac{r\varepsilon}{4} n$$



Recap

- bootstrapping ave. deg to min deg cond.
- Induction (on the # parts) to build

  $K_{t,\dots,t}$

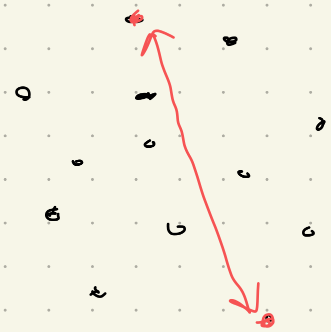
- Claim Many vs (Z) have high deg to each X_i

  Double counting

- Pigeonhole $Z \,\&\, \overbrace{X_1,\dots,X_s}^{I.H.} \Rightarrow X_1,\dots,X_s, \overset{\overset{Z}{\cup\!|}}{X_{s+1}}$

• Application of Turán's thm

$\mathbb{R}^2$



Thm (Erdős) $\forall$ arrangement P of n pts in $\mathbb{R}^2$ w/ diameter 1.

$\Longrightarrow$ # pairs of pts in P w/ dist. $> 1/\sqrt{2}$ $\leq \dfrac{n^2}{3}$

Pf Let G be an n-vx aux. gr w/ $V(G) = P$

$x \sim y \iff \text{dist}(x, y) > 1/\sqrt{2}$

suffices to show

STS $e(G) \leq n^2/3$

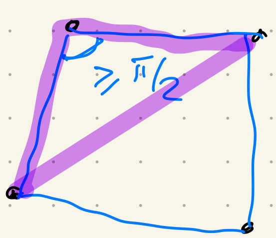
By Turán's thm, we need to prove

G is K_4-free.

Suppose $\exists K_4 \subseteq G$

4 pts in the plane

①



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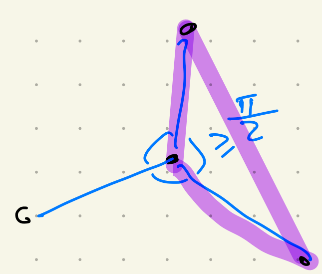
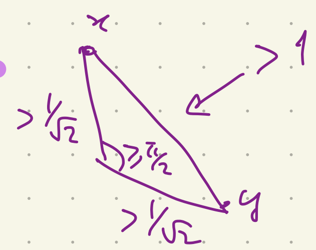


$x, y \in P$

$\text{dist}(\ \ ) > 1$ ↯



In either case $\exists$ angle $\geq \pi/2$

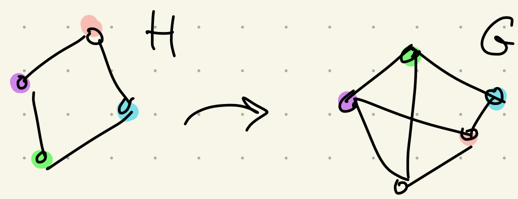
Pf 2 of Erdős-Stone-Simonovits

Modern pf using an embedding lem (from Sz. Reg. Lem)

Use it as a black box for now

- conceptually simpler than pf 1.

Recall subgr



injective map preserving adjacencies

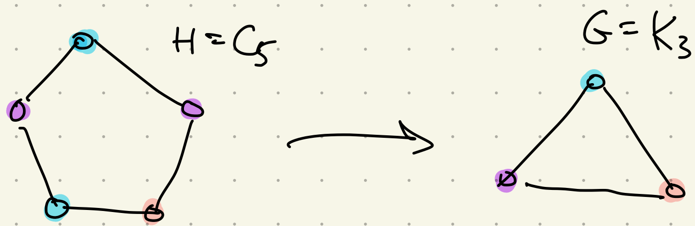
Def (Homomorphism)

A homomorphism from H to G is a map

$\varphi : V(H) \to V(G)$ preserving the adj. i.e.

$$\forall\, uv \in E(H) \Rightarrow \varphi(u)\varphi(v) \in E(G)$$

Ex



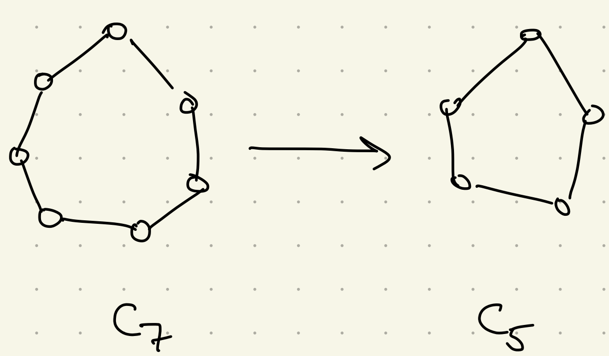
$C_5 \not\subseteq K_3$

Rmk $\exists$ homomorphism $\varphi : H \to K_r$

$\iff \chi(H) \le r$

← Exer

Exer



1) find a hom. $C_7 \to C_5$

2) Prove that $\nexists$ hom. from $C_5 \to C_7$

If $\exists$ hom. $\varphi : H \to G$, we say that G is a homomorphic image of H.

<u>Ex</u> $K_{\chi(H)}$ is a homomorphic image of H.

<u>Def</u> edge density of an n-vertex graph G: $\dfrac{e(G)}{\binom{n}{2}} = \tau$

- the pecentage of pairs of vxs that are adj in G
  - prob: unif at random choose two vxs u, v in G. $\mathbb{P}(uv \in E(G)) = \tau$

<u>Embedding lem</u>: Let $\forall \varepsilon > 0$ $\forall H$ be a graph, then the following holds for all suff. large n. $n \geq n_0(\varepsilon, H)$

$\forall$ n-vx H-free graph G with edge density $\tau > 0$

$\Rightarrow \exists$ a (reduced) graph $R = R(\varepsilon)$ with

$$\begin{cases} \text{edge density of } R \geq \tau - o_R(1) \quad \geq \tau - \varepsilon \\ R \text{ contains no homomorphic image of } H \end{cases}$$

<u>Why embedding lem useful</u>.

$G \to R$

Without lossing much on edge density, we bootstrap H-freeness to H-hom-freeness

- $\nexists$ hom $H \to R$

$\Updownarrow$

- R is H-hom-free

Pf 2 of E-S-S

$\left[\begin{array}{l} G:\ H\text{-free},\quad \chi(H) = r+1 \\ \Rightarrow e(G) \le \left(1 - \frac{1}{r} + \varepsilon\right)\frac{n^2}{2} \end{array}\right.$

$\Updownarrow$

G: H-free

τ = edge density of $G \le 1 - \frac{1}{r} + o(1)$

Upp bd.

- Embedding Lem on G

$\Rightarrow \exists R$ $\left\{\begin{array}{l} \text{edge density of } R \ge \tau - o(1) \\ H\text{-hom-free} \Rightarrow K_{r+1}\text{-free} \end{array}\right.$

$\boxed{\begin{array}{l} K_{r+1}\text{-free} \\ e(R) \le \left(1 - \frac{1}{r}\right)\frac{|R|^2}{2} \end{array}}$

- Apply Turán on R $\Rightarrow$

$\tau - o(1) \le$ edge density of $R \le 1 - \frac{1}{r} + o(1)$

Rmk Embedding Lem reduces ESS to Turán's thm.

§ Supersaturation

ESS : $ex(n,H) = \left(1 - \frac{1}{\chi(H) - 1} + o(1)\right)\frac{n^2}{2}$

Q: What happens when we have more edges than this threshold?

By ESS $\Rightarrow$ at least one copy of (the forbidden graph) H would appear is G

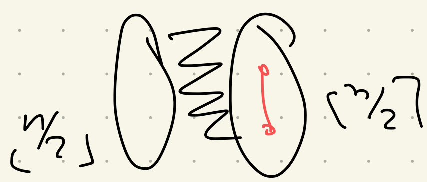
In fact, we shall see that "many" copies emerge. Such phenomenon is called "Supersaturation"

Example

Thm $\forall$ n-vx gr G, if $e(G) \geq \lfloor \frac{n^2}{4} \rfloor + 1 \implies$ G contains $\geq \lfloor \frac{n}{2} \rfloor$ Δs

$\uparrow$ $ex(n, K_3)$

Rmk: Tight:



- We consider asymp. ver: G w/ edge density above ESS.

Def: Given a graph H, its Turán density is

$$\pi(H) = \lim_{n \to \infty} \frac{ex(n, H)}{\binom{n}{2}}$$

$|||$ max edge density of an n-vx H-free gr G.

Q: Is it well-defined?

- ESS: $\forall$ H, $\pi(H) = 1 - \frac{1}{\chi(H) - 1}$

Thm $\forall H$, $\pi(H)$ exists.

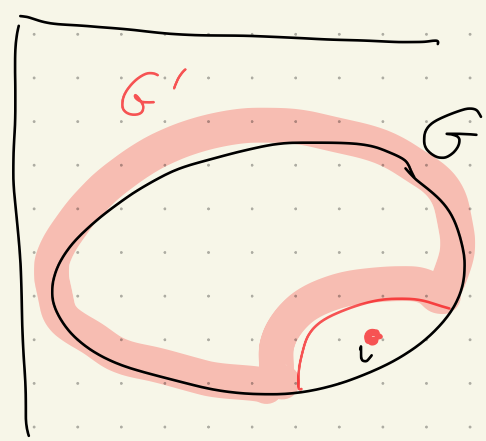
$$\pi(H) = \lim_{n\to\infty} \frac{ex(n,H)}{\binom{n}{2}}$$

Idea: By random sampling

Pf: Let $0 \le \alpha_n = \frac{ex(n,H)}{\binom{n}{2}} \le 1$, $\pi(H) = \lim_{n\to\infty} \alpha_n$

STS this bdd seq $\{\alpha_n\}_{n\in\mathbb{N}}$ is non-increasing

Goal: $\alpha_n \le \alpha_{n-1}$



- Take an extremal n-vx H-free G

  with $e(G) = ex(n,H)$

  $$\alpha_n = \frac{e(G)}{\binom{n}{2}}$$

- Unif. at random, delete a vx v from G

$$\Rightarrow G' = G - v$$

$\nwarrow$ random $(n-1)$-vx subgr of G

Fix an edge $e \in E(G)$

$$e \in E(G') \iff v \notin e \quad \Leftarrow \Pr(v\in e) = \frac{2}{n}$$

$$\Rightarrow \Pr(e \in E(G')) = 1 - \frac{2}{n}$$

Linearity of Expectation

$$\Rightarrow \mathbb{E}\, e(G') = \sum_{e\in E(G)} \Pr(e \in E(G'))$$

$$= e(G) \cdot \left(1 - \frac{2}{n}\right)$$

$$= \alpha_n \cdot \binom{n}{2}\left(1 - \frac{2}{n}\right)$$

$$= \alpha_n \cdot \binom{n-1}{2}$$

First Moment Method

$\Rightarrow$ $\exists$ a choice of v s.t.

$(-) \cdots \quad e(G') \geq \mathbb{E}\, e(G') = \alpha_n \cdot \binom{n-1}{2}$

- $G' \leq G \Rightarrow G'$ is still H-free

$(=) \cdots \Rightarrow \quad e(G') \leq ex(n-1, H) = \alpha_{n-1} \cdot \binom{n-1}{2}$

$(-)$ & $(=)$ $\Rightarrow$ $\alpha_n \leq \alpha_{n-1}$

HW1 up on homepage

due March/28