The codegree Turán density

Def: Given a k-graph H and $S \subseteq V(H)$, the degree of S is $d(S) = |\{e \in E(H) : S \subseteq e\}|$.
For $i \in [k-1]$, we set $\delta_i(H) = \min_{S \in V(H)^{(i)}} d(S)$ and call it the minimum i-degree of H. For $i = k-1$, $\delta_{k-1}(H)$ is also called the minimum codegree of H.

Def: Given a k-graph F and $n \in \mathbb{N}$, we define $ex_{k-1}(n, F) = ex_{co}(n, F) = \max\{\delta_{k-1}(H) : H$ is an F-free k-graph on n vertices$\}$

Def+Lemma (Mubayi, Zhao): The limit $\lim_{n \to \infty} \frac{ex_{k-1}(n, F)}{n}$ exists. It is called the codegree Turán density of F and we denote it by $\gamma(F)$.

We omit the proof here.

Thm (Mubayi): $\gamma(\mathbb{F}) = \frac{1}{2}$

Proof: For the lower bound, it is enough to note that the balanced "bipartite" 3-graph (two vertex classes with all those triples as edges which intersect both vertex classes) has minimum codegree $\sim \frac{n}{2}$ and does not contain $\mathbb{F}$.
For the upper bound, we use the following lemma.

Def: Let F_2 be the 3-graph obtained from the $K_4^{(3)-}$ by duplicating the apex, i.e.
$F_2 = (\{1,..,5\}, \{134, 135, 145, 234, 235, 245\})$.
Set $\mathcal{F}_2 = \{F_2\}$ and for $t>2$, let $\mathcal{F}_t$ be the family of 3-graphs obtained as follows. Take $F \in \mathcal{F}_{t-1}$, add two new vertices x and y and add t edges of the form axy, $a \in V(F)$.

Lemma (Mubayi, Rödl): For $F \in \mathcal{F}_t$, we have
1) F has $2t+1$ vertices;
2) Every set of $t+2$ vertices in F spans at least one edge;
3) $\pi(\mathcal{F}_t) \leq \frac{1}{2}$ for $t \geq 2$.

Let us first use this lemma to prove the theorem.

Proof of thm: Given $\varepsilon > 0$, choose $n_0, t \in \mathbb{N}$ with $n_0^{-1} \ll t^{-1} \ll \varepsilon$. Let H be a 3-graph on $n \geq n_0$ vertices with $\delta_{k-1}(H) \geq (\frac{1}{2}+\varepsilon)n$. Recall that by supersaturation, we have $\pi(\mathcal{F}_t) = \pi(\mathcal{F}_t')$, where $\mathcal{F}_t'$ is the family of 3-graphs obtained from $\mathcal{F}$ by replacing each $F \in \mathcal{F}$ with $F(2)$. Further note that $e(H) \geq \frac{1}{3}\binom{n}{2}(\frac{1}{2}+\varepsilon)n > (\frac{1}{2}+\varepsilon)\binom{n}{3}$, whence we conclude that $F(2) \subseteq H$ for some $F \in \mathcal{F}_t$, using Part 3) of the lemma. Let $V(F(2)) = X \cup X'$ such that every vertex in F has one copy in X and one in X'. For every $x \in X$, denote its copy in X' by x'. Further, let $N(x,x') = \{z \in V(H) : xx'z \in E(H)\}$ be the coneighbourhood of x and x'. We then have

$$\sum_{x \in X} |N(x,x')| \geq (2t+1)(\tfrac{1}{2}+\varepsilon)n,$$

where we used Part 1) of the lemma and

the minimum codegree condition on H. Thus, there is some $y \in V(H)$ with $y \in N(x,x')$ for at least $(\frac{1}{2}+\varepsilon)(2t+1) \geq t+2$ different choices of $x \in X$. By Part 2) of the lemma, there are $a,b,c \in X$ with $abc \in E(H)$ and $y \in N(x,x')$ for all $x \in \{a,b,c\}$. Now $H[\{a,b,c,a',b',c',y\}]$ contains a copy of $\mathbb{F}$.

et us now prove the lemma.

roof of lemma:

Part 1): This is an easy induction.

Part 2): We proceed by induction on t. For $t=2$, it is easy.

$t-1 \to t$: Let $F' \in \mathcal{F}_t$ be obtained by taking $F \in \mathcal{F}_{t-1}$, adding vertices x and y, and adding t edges axy with $a \in V(F)$. Now consider a set S of $t+2$ vertices. If $|S \cap V(F)| \geq t+1$, we get an edge in S by induction. Otherwise, $|S \cap V(F)| = t$ and $x,y \in S$, and since $|V(F)| = 2t-1$, there is some $a \in S$ with $axy \in E(F$

Part 3): We commence by recalling $\pi(K_4^{(3)-}) \leq \frac{1}{3}$, which implies $\pi(F_2) = \pi(\mathcal{F}_2) \leq \frac{1}{3}$ using supersaturation. Hence, there is some constant $c > 2$ such that $ex(n, \mathcal{F}_2) \leq \frac{1}{2}\binom{n}{3} + 2cn^2$ for all $n \in \mathbb{N}$. Now we argue by induction on t that
$ex(n, \mathcal{F}_t) \leq \frac{1}{2}\binom{n}{3} + tcn^2$ for all $n \in \mathbb{N}$.

$t-1 \to t$: Use induction on n. For $n = 2t+1$, this is easy to see.

$n-1 \to n$:

Let H be a 3-graph on n vertices with $e(H) \geq \frac{1}{2}\binom{n}{3} + tcn^2$ edges. By the induction on t, we know that $F \subseteq H$ for some $F \in \mathcal{F}_t$. Let L be the multigraph on $V(H) \setminus V(F)$ whose edges are given by the edges in the links of the vertices in F. By definition of $\mathcal{F}_t$, if any edge in L has multiplicity $\geq t$, this pair together with F forms a copy of an element of $\mathcal{F}_t$ in H. If this does not happen, there is a vertex $v \in V($
with

$$d_H(v) \leq \underbrace{\frac{t-1}{2t-1}\binom{n-2t+1}{2}}_{\substack{1 \text{ vtx in } F, \text{ two} \\ \text{in } V(H) \setminus V(F)}} + \underbrace{(2t-2)(n-2t+1)}_{\substack{2 \text{ vts in } F, \text{ one in} \\ V(H) \setminus V(F)}} + \underbrace{\binom{2t-1}{2}}_{\substack{3 \text{ vts} \\ \text{in } F}}$$

$$< \frac{t-1}{2t-1}\binom{n-1}{2} + 2tn$$

Deleting v leaves us with a 3-graph on $n-1$ verti
with at least

$$e(H) - \frac{t-1}{2t-1}\binom{n-1}{2} - 2tn \geq \frac{1}{2}\binom{n}{3} + tcn^2 - \frac{1}{2}\binom{n-1}{2} - 2tn$$

$$> \frac{1}{2}\binom{n-1}{3} + ct(n-1)^2$$

edges. By induction, this 3-graph contains an element of $\mathcal{F}_t$.