Lecture 3

<u>Last time</u>

- E–Sz, pigeonhole
- Sperner via LYM ineq, double/prob. counting
- $ex(n, H)$
- Turán thm: $ex(n, K_{r+1}) \leq \left(1 - \frac{1}{r}\right)\frac{n^2}{2}$

  Furthermore, $T_{n,r}$ is the unique extremal graph.

  Pf 3. Zykov Symm.

—— Motzkin - Straus Symm. ——

Viewed as a continuous ver. of Zykov's Symm.

- G graph   symmetric 0/1-matrix   adjacency matrix

$V = [n]$

$$(a_{ij})_{i,j\in[n]} = A_G = \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ i \\ \vdots \\ n \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & \cdots & & j & \cdots n \end{array} \\ \begin{pmatrix} 0 & & & & \vdots & \\ & 0 & & & \vdots & \\ \cdots & & 0 & \cdots & 1 & \\ & & & \ddots & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} \end{array} \qquad a_{ij} = 1 \iff ij \in E$$

Consider the quadratic form $\quad \underline{x} \in \mathbb{R}^n$

$$\lambda_G(\underline{x}) = \frac{1}{2} x^T A_G x = \frac{1}{2} \sum_{i,j\in[n]} x_i x_j \cdot a_{ij} = \sum_{ij\in E(G)} x_i x_j$$

Lagrangian of G

$$\lambda(G) = \sup_{x \in \Delta^{n-1}} \lambda_G(x)$$

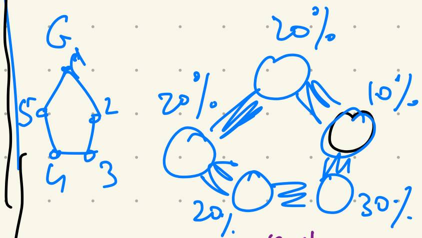
$$\Delta^{n-1} = \left\{ \underline{x} \in \mathbb{R}^n : \quad x_i \geq 0 \quad \forall i \in [n] \quad \text{and} \quad \sum_{i=1}^{n} x_i = 1 \right\}$$

Simplex

basically ... # edges in a weighted ver. of G

$$\lambda_G(\underline{x}) = \sum_{ij \in E(G)} x_i x_j$$

$\underline{x} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$ $\quad \lambda_G(\underline{x}) = e(G)$



$G_{\underline{x}}$ $\quad x = \begin{pmatrix} 1/5 \\ 1/10 \\ 3/10 \\ 1/5 \\ 1/5 \end{pmatrix}$

an $x \in \Delta^{n-1}$ corresponds to a weighted ver. of G

$$\lambda_G(\underline{x}) = e(G_{\underline{x}})$$
$x \in \Delta^{n-1}$

<u>Thm</u> (Motzkin - Straus)

$\forall$ n-vx G $\quad K_{r+1}$-free

$\forall$ $\underline{x} \in \Delta^{n-1}$

$\Rightarrow \exists\, y \in \Delta^{n-1}$ s.t.

- $\lambda_G(y) \geq \lambda_G(x)$
- $\text{supp}(y)$ induces a clique in G

the set of non-zero coord. of y

In particular, $\quad \lambda(G) = \frac{1}{2}\left(1 - \frac{1}{r}\right)$

$\underline{x} = \left(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}\right)^T$

$\lambda_G(\underline{x}) = \dfrac{e(G)}{n^2}$

$\leq \lambda(G)$

$= \frac{1}{2}\left(1 - \frac{1}{r}\right)$

<u>Idea</u>: 'mass transportation' if $\underline{x}$ has mass on two coord. $\Leftrightarrow$ two vxs not adjacent

$\Rightarrow$ move weight from one coord. to another without decreasing $\lambda_G(\cdot)$

Pf: Take $y \in \Delta^{n-1}$ with minimal support s.t.

$$\lambda_G(y) \geq \lambda_G(x)$$

We shall prove that $\mathrm{supp}(y)$ induces a clique.

Suppose $\exists$ two vxs not adjacent, say $\{1,2\} \notin E(G)$

$$y \longrightarrow y+z, \quad z = \begin{pmatrix} z \\ -z \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

move weight from vx 2 to vx 1 (amount z)

$$\lambda_G(y+z) = \frac{1}{2}(y+z)^T A (y+z)$$

$$= \frac{1}{2} y^T A y + \frac{1}{2} \cdot 2 \cdot z^T A y + \underbrace{\frac{1}{2} z^T A z}_{= \sum_{ij \in E} z_i z_j = 0}$$

$$= \lambda_G(y) + z^T A y$$

linear funct. in z

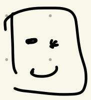
linear $\Rightarrow$ we can choose z s.t.

$$\lambda_G(y+z) \geq \lambda_G(y) \geq \lambda_G(x)$$

$$y+z \in \Delta^{n-1}$$

and $\mathrm{supp}(y+z) \subsetneq \mathrm{supp}\, y$

$\lightning$ minimality of $\mathrm{supp}(y)$



Exer ver.

- Local Turán (Bradac, Malec-Tompkins)

$\forall$ n-vx $G \Rightarrow \displaystyle\sum_{e \in E(G)} \frac{k(e)}{k(e)-1} \leq n^2/2$

$k(e)$ = size of largest clique in G containing the edge e.

The 'in particular' part amounts to proving the following

Exer $\lambda(K_r) = \frac{1}{2}\left(1 - \frac{1}{r}\right)$

Pf 5 Prob. pf by Caro-Wei

Thm $\forall G$, $\alpha(G) \geq \sum_{v \in V(G)} \frac{1}{d(v)+1}$

- First Moment Method -

r.v. X

$\mathbb{E}X = k \Rightarrow \exists$ outcome of X taking value $\leq k$ (resp. $\exists$ one w/ value $\geq k$)

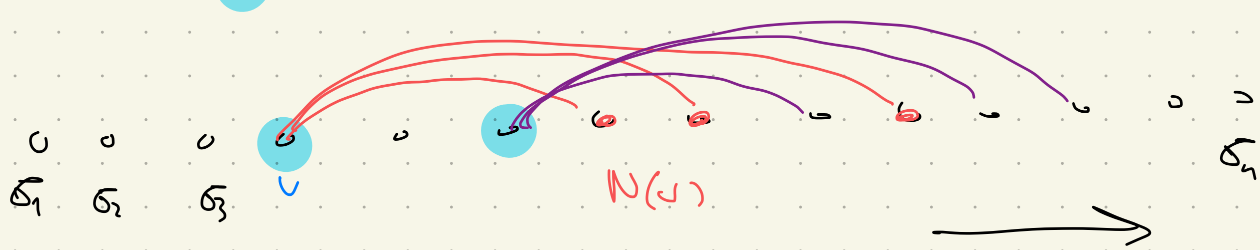
Def Independent set

$S \subseteq V(G)$ if $e_G(S) = 0$ S induces no edge.

Indep. # of G, $\alpha(G)$, = size of largest indep. set in G

Pf • Consider a unif. random perm. σ of $V(G)$

Define $I = \{v \in V(G) : v \text{ precedes all } u\text{s in } N(v)\}$



Obs I is an indep. set

Suffices to show $\mathbb{E}|I| = \sum_{v \in V(G)} \frac{1}{d(v)+1}$ as

then $\exists I$ s.t. $\alpha(G) \geq |I| \geq \mathbb{E}|I|$

$$\mathbb{E}|I| = \sum_{v\in V(G)} \Pr(v\in I) = \sum_{v\in V(G)} \frac{1}{d(v)+1}$$

σ unif random ordering

Erdős - Stone thm

Fundamental thm in extremal graph theory

Turán $\Rightarrow$ $\quad ex(n, K_{r+1}) = \left(1-\frac{1}{r}\right)\frac{n^2}{2} \pm O(r)$

Thm (Erdős-Stone 46)

$\forall H, \quad ex(n,H) = \left(1 - \frac{1}{\chi(H)-1} \pm o(1)\right)\frac{n^2}{2}$

$\forall \varepsilon, \forall H \; \exists \overset{n_0 \geq}{n_0(\varepsilon, H)}$ TFH $\forall n \geq n_0$

$$ex(n,H) = \left(1 - \frac{1}{\chi(H)-1} \underline{\pm \varepsilon}\right)\frac{n^2}{2}$$

$x = a \pm b$

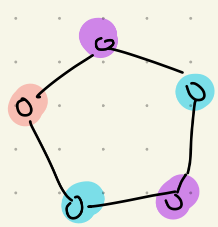
$a-b \leq x \leq a+b$

Def gr. H, its chromatic # $\chi(H)$

is the min # k s.t. we color vxs of H by k colors

so that no adj vxs receive the same color.

C_5



$\chi(C_5) = 3$

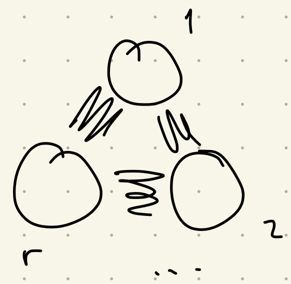
Rmk · E-S determines asym. $ex(n,H)$ for all non-bip. H.

· For bip. H : ONLY get $o(n^2)$

$\chi(H) = r+1$

Lower bound of E-S : find n-vx H-free w/ $\geq \left(1-\frac{1}{r}-\varepsilon\right)\frac{n^2}{2}$ edges

$H \notin T_{n,r}$



Upper bound NTS $\forall$ n-vx G w/ $\geq \left(1-\frac{1}{r}+2\varepsilon\right)\frac{n^2}{2}$ edges $\Rightarrow$ $H \subseteq G$

$t = |H| = \#$ vxs of H

$$H \underset{\chi(H)=r+1}{\subseteq} K_{\underbrace{t,\dots,t}_{r+1 \text{ parts}}} \subseteq G$$

will prove this

Induction on r.

Lem $\forall\ 0<\varepsilon<c<1$, $\exists\ n_0 = n_0(\varepsilon, c)$ s.t. $\forall\ n \geq n_0$ TFH.

$\forall$ G n-vx w/ $e(G) \geq cn^2/2 \Rightarrow \exists\ G' \subseteq G$ on $n' \geq \sqrt{\varepsilon}n/2$ vxs s.t. $\delta(G') \geq (c-\varepsilon)n'$

Pf Idea (Exer) Keep removing vxs w/ low deg from G $\rightsquigarrow$ G' desired $\square$

Pf (E-S) Lem $\Rightarrow$ $G' \subseteq G$ on $n' \geq \frac{\sqrt{\varepsilon}\, n}{2}$ vxs

(so n' is suff. large as long as n is suff. large)

(Rewrite G for G')

Start w/ n-vx G w. $\delta(G) \geq \left(1 - \frac{1}{r} + \varepsilon\right) n$

$\boxed{\text{Goal}}$ embed $K_{\underbrace{t, \dots, t}_{r+1 \text{ parts}}} \subseteq G$

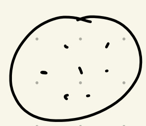
Induction on # parts $1 \leq s \leq r+1$ that

$\forall\, t,\ \exists\, n_0 = n_0(t, s)$ s.t.

$\forall\, n \geq n_0\ \forall\, G$ n-vx w/ $\delta(G) \geq \left(1 - \frac{1}{r} + \varepsilon\right) n$

we have $K_{\underbrace{t, \dots, t}_{s \text{ parts}}} \subseteq G$

- Base case : $s = 1$



empty gr on t vxs

trivial

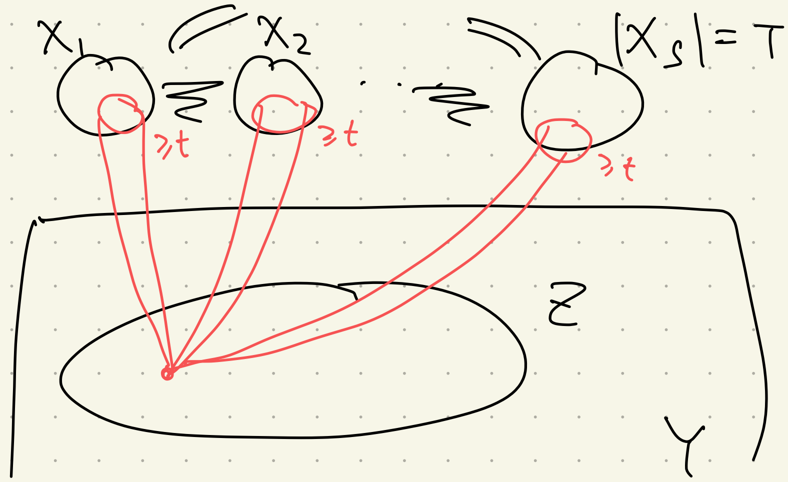
- Inductive step : suppose true for all $s \leq r$

Need to embed $K_{\underbrace{t, \dots, t}_{s+1 \text{ parts}}} \subset G$

By I.H. $\Rightarrow \exists\, K_{\underbrace{T, \dots, T}_{s \text{ parts}}} \subseteq G$ where $T = \frac{4}{r \cdot \varepsilon} \cdot t$

on partite sets say $X_1, \dots, X_s$

$Y = V(G) \setminus \left(\bigcup_{i=1}^{s} X_i\right)$



Claim: Let $Z \subseteq Y$ be the set of all vxs with $\geq \frac{r\varepsilon}{4}|X_i| = t$ neighbors in each X_i, $i \in [s]$

$\Longrightarrow \quad |Z| \geq \frac{r\varepsilon}{4} \cdot n$

- [Claim $\Rightarrow$ Thm] each vx in Z gives rise to a copy of $K_{1,\underbrace{t,t,\dots,t}_{s \text{ parts}}}$ w/ the size-t part in each X_i

- \# choices for t-sets $X_i' \subseteq X_i = \binom{T}{t}^s$

- pigeonhole $\Rightarrow \quad \geq \dfrac{|Z|}{\binom{T}{t}^s} \geq \boxed{\dfrac{r\varepsilon}{4\binom{T}{t}^s} \cdot n \geq t}$

vxs in Z sharing the same ....

$\Longrightarrow \quad K_{\underbrace{t,\dots,t}_{s+1 \text{ parts}}}$

Rmk $\quad (1 - \frac{1}{r} + \varepsilon)\frac{n^2}{2}$ edges $\Longrightarrow$ $\Omega(\log n)$ blow up of K_{r+1}