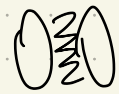
Lecture 2

Recap

- graphs paths/cycles / complete (multipartite) gr
- $\sum_{v \in V} d(v) = 2e(G)$ Handshaking lemma
- Mantel's thm : G n-vx Δ-free $\Rightarrow e(G) \le \lfloor \frac{n^2}{4} \rfloor$



$\lceil n/2 \rceil \quad \lfloor n/2 \rfloor$

Induction (on # vxs n)

- Ramsey # lower bd constr. using probability
- odd/even town via linear algebra

—— Erdős - Szekeres ——

Consider a seq of natural numbers

Ex : $k=3$

10, 5, 7, 4, 6

WANT : long monotone subseq (↗ , ↘)

Thm (E-Sz) $\forall$ seq of distinct natural #s of size $(k-1)^2+1 \Rightarrow \exists$ monotone seq of size k

Idea : pigeonhole / averaging argument.

Pf: Consider an arbitrary seq $a_1, a_2, \dots, a_{(k-1)^2+1}$, $a_i \in \mathbb{N}$

Goal: find monotone $\nearrow$ or $\searrow$ $a_{i_1}, \dots, a_{i_k}$

☆ Label each # a_i by the length of longest monotone $\nearrow$ seq ending at a_i : $\ell(a_i)$

Obs 1
- $1 \le \ell(a_i) \le k-1$ o.w. ☺

a_i	10	5	7	4	6
$\ell(a_i)$	1	1	2	1	2

Obs 2
- By pigeonhole & Obs 1

$\exists$ some label $r \in [k-1]$ s.t. $\exists$ $a_{i_1}, \dots, a_{i_k}$ s.t.

$\ell(a_{i_j}) = r \quad \forall j \in [k]$

Obs 3
- $a_{i_1} > a_{i_2} > \dots > a_{i_k}$ $\Leftarrow$ form a monotone $\searrow$ seq

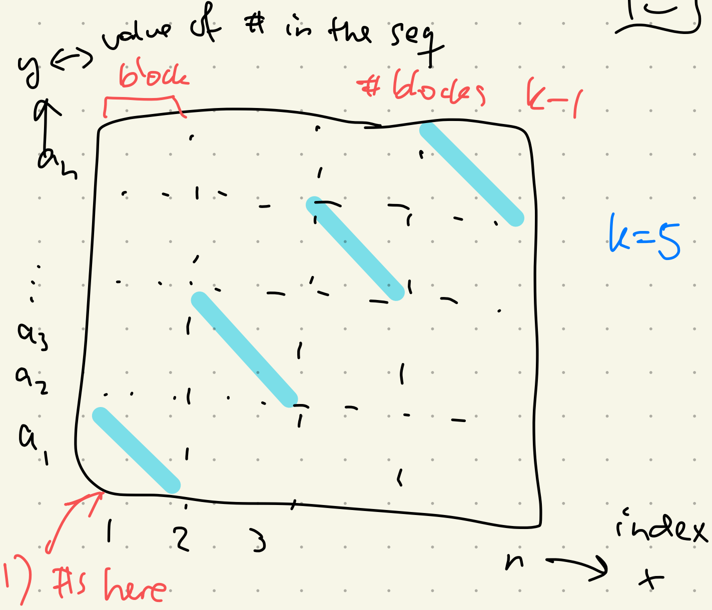
$\ell(\cdot)$ r r r

$a_{i_1} > a_{i_2}$ for o.w. $\ell(a_{i_2}) \ge r+1$ using ......

Rmk:
- Bound is optimal

Tight example:

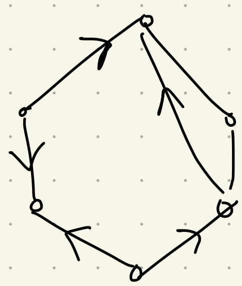
$\exists$ seq of size $(k-1)^2$ without $\nearrow$ or $\searrow$ seq of size k



- asym ver. $\forall$ seq size $(a-1)(b-1)+1$

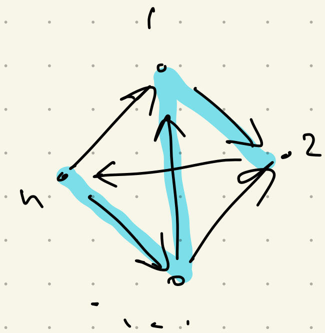
  $\Rightarrow$ $\exists \nearrow$ size a or $\searrow$ size b.

- directed graph



Def: tournament

An orientation of complete graph



Prop (Exer) $\forall$ tournament of order n,

$\exists$ HAM path (a path that visited all vxs)

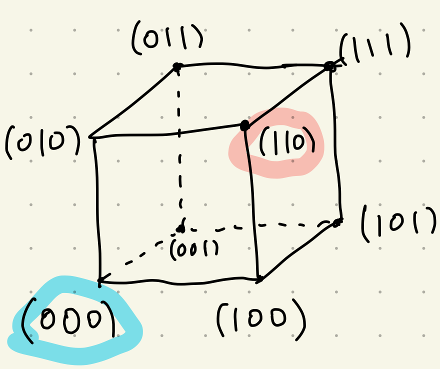
a directed

——— Sperner's thm ———

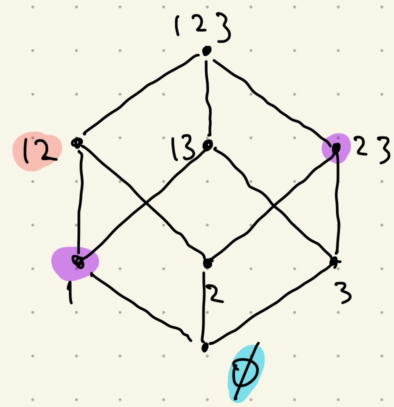
Consider $2^{[n]}$ (power set of $[n]$)

$\{0,1\}^n$ hypercube : $V = \{0,1\}^n$ E : $x \sim y$ iff exactly they differ at 1 place.

Ex $n=3$



family of all subsets of $[n]$



antichain

Def A family of sets $\mathcal{A} \subseteq 2^{[n]}$ is an antichain if no element of $\mathcal{A}$ is a subset of another.

Ex Any level (all sets of the same size) ...

$$\binom{n}{k} = \# \text{ all size-}k \text{ sets} \leq \binom{n}{\lfloor n/2 \rfloor}$$

Thm (Sperner) The size of the largest antichain in $2^{[n]}$ is $\binom{n}{\lfloor n/2 \rfloor}$

LYM ⇒ sperner

$$1 \geq \sum \frac{1}{\binom{n}{|F|}} \geq \sum \frac{1}{\binom{n}{\lfloor n/2 \rfloor}} = |\mathcal{F}| / \binom{n}{\lfloor n/2 \rfloor}$$

Thm (LYM ineq) Lubell – Yamamoto – Meshalkin

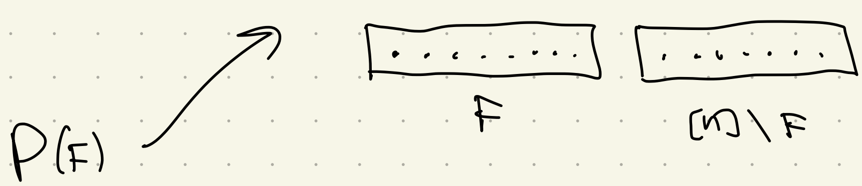
$\forall$ antichain $\mathcal{F} \subseteq 2^{[n]}$

$$\Rightarrow \sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{|F|}} \leq 1$$

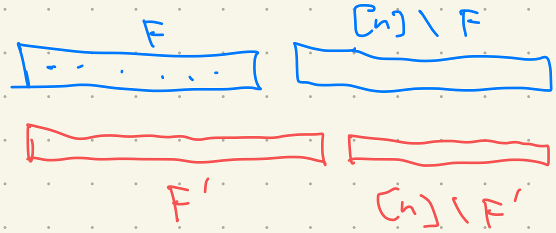
- Induction Mantel
- Pigeonhole E-Sz
- Double counting LYM

Pf 1 (Double counting)

- $n! = \#$ of all permutation of $[n]$
- $F \in \mathcal{F}$ # permutations satisfying $= |F|!\,(n-|F|)!$



Obs $\quad P(F) \cap P(F') = \emptyset \quad \forall F \neq F' \in \mathcal{F}$



o.w. $F \subset F'$ or $F' \subset F$.

$$\sum_{F\in\mathcal{F}} |P(F)| = \sum_{F\in\mathcal{F}} |F|!\,(n-|F|)! \le n!$$

Pf 2

- Take a unif random perm.

$$\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) \sim S_n$$

- $\forall k \in [n]$, let $I_k = \{\sigma_1, \sigma_2, \ldots, \sigma_k\}$ initial k-segment

$$I_1 \subset I_2 \subset I_3 \subset \cdots \subset I_n$$

$\Rightarrow$ at most one of them can be in $\mathcal{F}$

- Note $I_k \sim \binom{[n]}{k}$ ← all k-subsets of [n]

\# of k-sets in $\mathcal{F}$

$$1 \ge \quad \Rightarrow \mathbb{E}\, \# I_k \text{ in } \mathcal{F} = \sum_{k=1}^{n} \frac{f_k}{\binom{n}{k}} = \sum_{F\in\mathcal{F}} \frac{1}{\binom{n}{|F|}}$$

linearity of expectation

$$\| \quad \sum_{k=1}^{n} \Pr(I_k \in \mathcal{F})$$

$\Pr(I_k \in \mathcal{F}) = \Pr(I_k = F \text{ for some } F \in \mathcal{F})$

Topic 1 Extremal Graph Theory

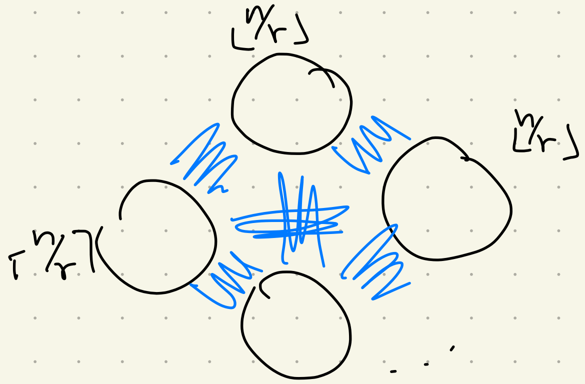
<u>Turán problem</u> G n-vxs, H-free. <u>Q</u>: max # edges in G ?

$$ex(n,H) = \max\{e(G) : G \text{ - } n \text{ vxs}, \text{ - } H\text{-free}\}$$

Turán/extremal number of H

Mantel: $ex(n,K_3) = \lfloor \frac{n^2}{4} \rfloor$

- Turán graph $T_{n,r}$



n-vertex complete ~~multipartite~~ r-partite graph w/ each part. set of size $\lfloor n/r \rfloor$ or $\lceil n/r \rceil$.

<u>Thm</u> (Turán) $ex(n, K_{r+1}) = e(T_{n,r})$

Furthermore, $T_{n,r}$ is the unique extremal graph.

<u>Fact</u> $\left(1-\frac{1}{r}\right)\frac{n^2}{2} - O(r) \le e(T_{n,r}) \le \left(1-\frac{1}{r}\right)\frac{n^2}{2}$

$O(\cdot)$, $o(\cdot)$, $\Omega(\cdot)$, $\omega(\cdot)$

$f(n) = o(g(n))$: $\frac{f(n)}{g(n)} \to 0$ $n \to \infty$

$f(n) = O(g(n))$: $f(n) \le C \cdot g(n)$

Exer Among all complete r-partite graphs on n vxs, $T_{n,r}$ has the most # edges.

- G n-vx K_{r+1}-free if $\exists$ r-partite graph F s.t. $e(F) \geq e(G)$

$\Rightarrow$ Turán's thm

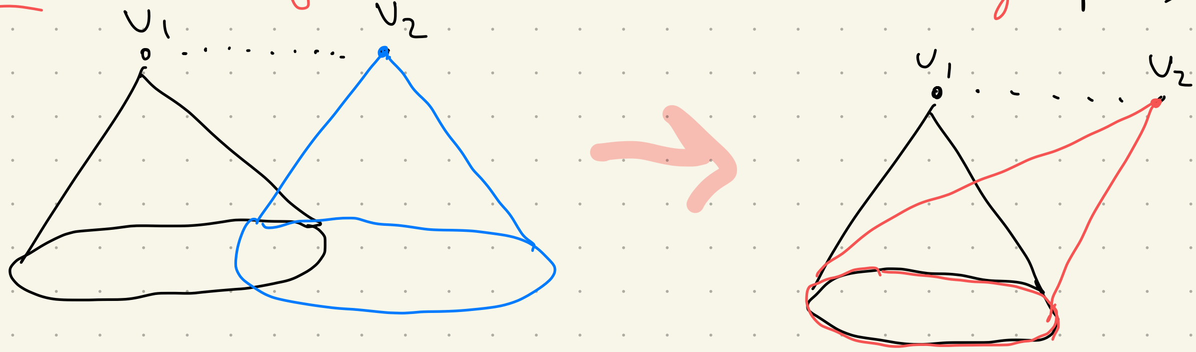
Exer Extend the pfs (Pf 1. Induction, Pf 2. Max-deg vx) of Mantel's thm to prove Turán's thm.

$\boxed{\text{Pf 3 Zykov's symmetrization}}$

Pf Consider an n-vx extremal gr. G: $e(G) = ex(n, K_{r+1})$

- Fix an ordering of vxs $v_1, v_2, \dots, v_n$
- Going along this ordering: symm non-adj pairs

Obs: non-adj pair of vxs have the same degree



If $v_1v_2 \notin E(G)$, make v_2 a twin of v_1

$1 \le i \le n$

- For v_i, find the smallest $j < i$ s.t. $v_i v_j \notin E(G)$, DO symm.

Process ends in n steps

At the end: non-adjacency is an equivalence relation

$\Rightarrow$ Obtain a complete multipartite gr.

\# parts $k \le r$

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March 3/12

10 AM Beijing time

11 AM Seoul time.