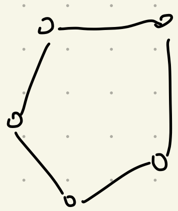
Lecture 1

群聊: Spring Course 2024

该二维码7天内(3月12日前)有效，重新进入将更新

- Graph (V, E)

  vertices ↑ ↖ edges (binary symm. on V)



5 vxs

5 edges

- Basic graph families

  - paths :

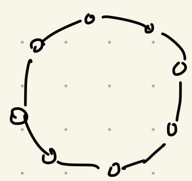
P_9

P_k

k ↖ # vxs in this path

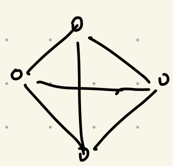
(order)

  - cycle :



C_8

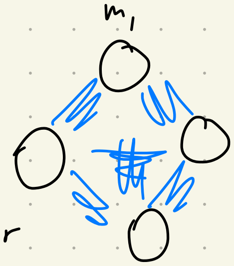
  - complete graph K_t



$t=4$

  - complete multipartite graph $K_{m_1, m_2, \dots, m_r}$

m_1

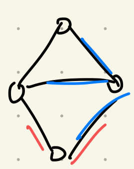


m_2

m_r

    - r parts
    - no edge in each part
    - all edges between different parts

  - degree of a vertex v : $d(v) = $ # edges incident to v



v ← deg = 3

u ← deg = 2

Prop $\forall G$, $2e(G) = \sum_{v \in V(G)} d(v)$

$e(G)$ ↑ # edges in G

- $\delta(G)$ min deg of G
- $\Delta(G)$ max deg of G

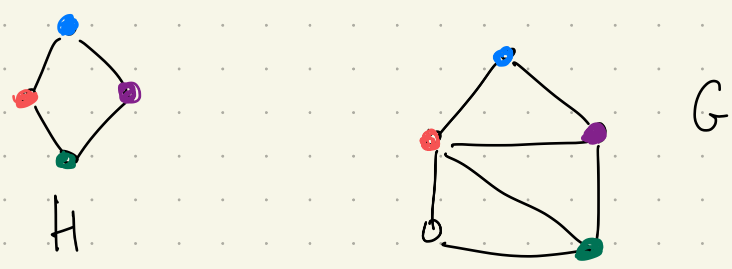
Def Subgraph containment

Given two graphs H & G, we say H is a subgraph of G, denoted by $H \subseteq G$, if

$\exists$ injective map $\varphi : V(H) \longrightarrow V(G)$ preserving adjacencies

i.e. $\forall\, uv \in E(H) \Rightarrow \varphi(u)\varphi(v) \in E(G)$

Ex



If no such map : G is H-free

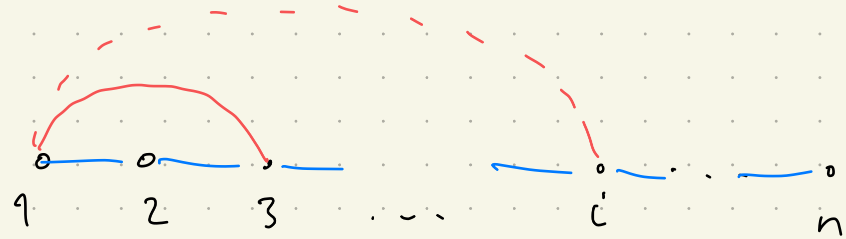
Prop $\forall G$ w/ average deg $d(G) = \sum_{v \in V(G)} d(v) \Big/ |V(G)|$

$$d(G) \geq 2 \implies \exists \text{ cycle} \subseteq G$$

Exer. Prove it

Exer Is this bound tight /optimal ? (YES)

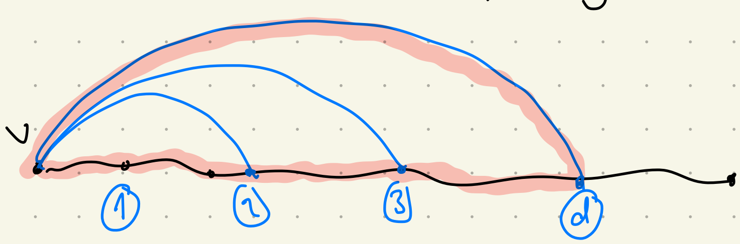
Rmk: We cannot say anything about this cycle (no control)



$d(G) = 2 \iff e(G) = n = |V(G)|$

Prop $\forall G$, $\delta(G) \geq d \Rightarrow \exists$ cycle of length $\geq d+1$ in G

Pf: • Take a longest path, say starting at some vertex v



longest $\Rightarrow$ $N(v)$ $\Rightarrow$ desired cycle

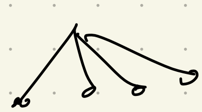
set of all neighbors of v

Rmk Bound optimal: Consider $G = K_{d+1}$

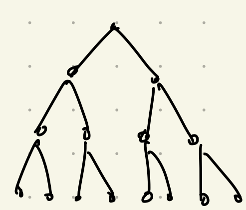
• (trees) A tree is a graph without any cycle (as a subgraph) (acyclic)

Ex. 

path

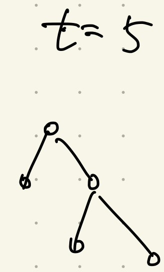


star



Ex $\forall$ tree on t vertices T

$\forall G \quad \delta(G) \geq t-1 \implies T \subseteq G$



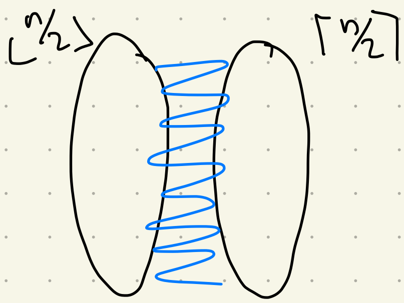
Rmk Optimal $G = K_t$

Mantel's thm

Thm (Mantel) $\forall G$ • Δ-free (K_3-free)
• n-vertex

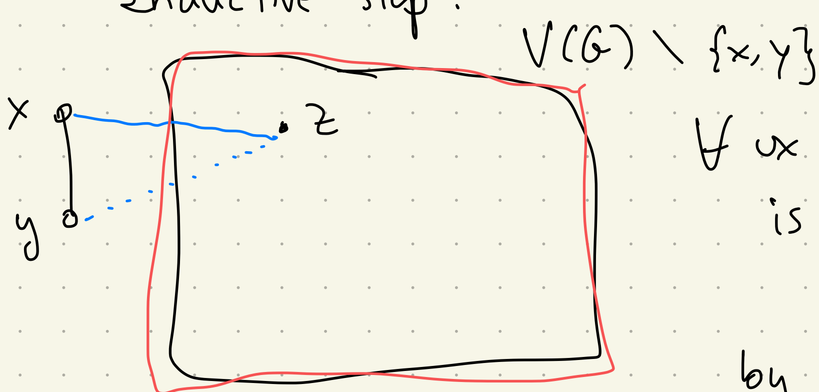
$$\implies e(G) \leq \left\lfloor \frac{n^2}{4} \right\rfloor$$

Rmk Optimal!



Pf 1 Induction on the number of vertices n.

- Base case : easy.
- Inductive step :



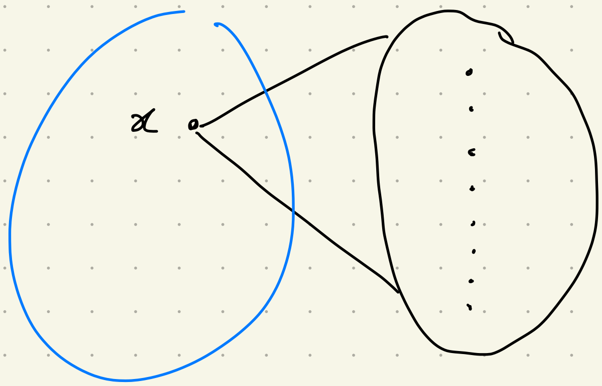
$\forall$ vx $z \neq x, y$ is adjacent to ≤ 1 vertex in $\{x, y\}$ by Δ-freeness

$$e(G) \le \underset{\uparrow \atop xy\text{ edge}}{1} + \underset{\uparrow \atop \text{edges from } V\setminus\{x,y\} \text{ to } \{x,y\}}{n-2} + \left\lfloor \frac{(n-2)^2}{4} \right\rfloor \le \left\lfloor \frac{n^2}{4} \right\rfloor.$$

$\uparrow$ I.H.

<u>Pf 2</u> (Max deg vx)

Consider a vertex of maximum deg $d(x) = \Delta(G)$



$Y = V \setminus X$

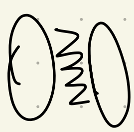
$X = N(v)$

- Δ-free

$\Rightarrow$ no edge in X

$\Rightarrow$ every edge in G is incident a vertex in Y

$$\Rightarrow \quad e(G) \le \sum_{y \in Y} d(y) \underset{\uparrow \atop \text{choice of } x}{\le} |X||Y| \le \left(\frac{|X|+|Y|}{2}\right)^2 = \frac{n^2}{4}$$



<u>Ex 1</u> Prove that $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ is the unique maximizer (extremal graph).

— Sets —

- Odd town even town problem

Setup : • A town of n ppl.

$\{1, 2, 3, \ldots, n\} =: [n]$

• form into clubs $C_1, C_2, \ldots$

$C_i \subseteq [n]$

Need $\forall$ distinct clubs C_i, C_j

$|C_i \cap C_j|$ even number

Q1 If each club C_i admits even # ppl,

then max # clubs there can be ?

A: We can have exponentially many.

Group ppl into pairs $\{1,2\}, \{3,4\}, \ldots \{\underset{\text{even}}{n-1, n}\}$

$\# \text{pairs} = n/2$

Clubs = all possible combinations of pairs

$2^{n/2}$

Q2 What if each club has odd size?

Try $C_i = \{i\}$ $\quad$ # clubs $= n$

$|C_i \cap C_j| = 0$ even

Thm $\forall \mathcal{F} \subseteq 2^{[n]}$ $\quad$ power set of $[n]$: family of all subsets of $[n]$.

① $\forall F \in \mathcal{F}$, $|F|$ odd

② $\forall$ distinct $F, F' \in \mathcal{F}$
$|F \cap F'|$ even

$\Longrightarrow |\mathcal{F}| \le n$. $\quad$ # sets in $\mathcal{F}$

Pf. Consider $\mathbb{1}_F$ for each $F \in \mathcal{F}$

① $\Rightarrow \langle \mathbb{1}_F, \mathbb{1}_F \rangle = 1$ (in $\mathbb{F}_2$)

② $\Rightarrow |F \cap F'| = \langle \mathbb{1}_F, \mathbb{1}_{F'} \rangle$
$= 0$ (in $\mathbb{F}_2$)

$n = 5$
$\{1, 2, 3, 4, 5\} = [n]$
$F = \{2, 4, 5\}$
$\mathbb{1}_F = (0, 1, 0, 1, 1)$

$\Rightarrow \{\mathbb{1}_F : F \in \mathcal{F}\}$ pairwise orthogonal
$\subseteq \mathbb{F}_2^n$

$\Rightarrow |\mathcal{F}| = |\{\mathbb{1}_F : F \in \mathcal{F}\}| \le \dim \mathbb{F}_2^n = n$ $\quad$ ☺

Ramsey

Philosophy: No complete disorder

∃ highly ordered substructure in any suff. large system.
large

Consider two edge-coloring of complete graphs.
blue/red

Goal: find large monochromatic complete subgraphs

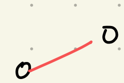
<u>Def</u> Ramsey number $R(s,t) = \min n \in \mathbb{N}$ s.t.
$s,t \in \mathbb{N}$

$\forall$ 2 edge-coloring of $K_n \Longrightarrow \exists$ either a blue K_s or a red K_t
blue/red

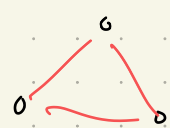
<u>Warm</u> $R(2,3) = 3$

Lower bound: by construction

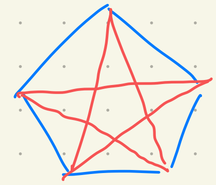
$R(2,3) > 2$



Upp bd $R(2,3) \le 3$

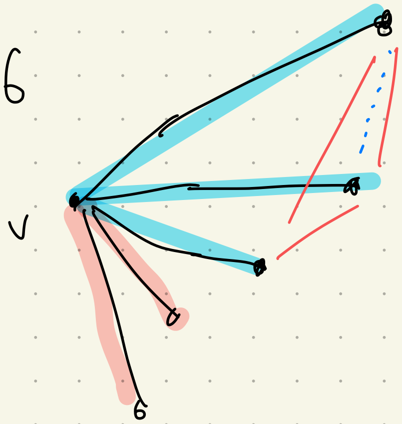


- $R(3,3) = 6$
- $R(3,3) > 5$



$R(3,3) \le 6$

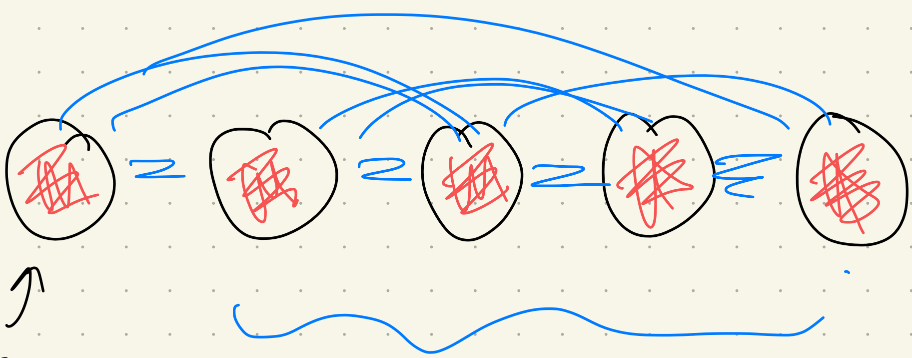
<u>Pf</u>



Pigeonhole $\Rightarrow$ ≥ 3 edges same color

Lower bound on $R(k,k) > n$: Need to find a 2-coloring of K_n w/ no red/blue K_k.

<u>Ex</u> $R(k,k) > (k-1)^2$



each gp size $k-1$

$\#$gps $= k-1$

total # vxs $= (k-1)^2$

- Use probabilistic method to prove lower bound on $R(k,k)$

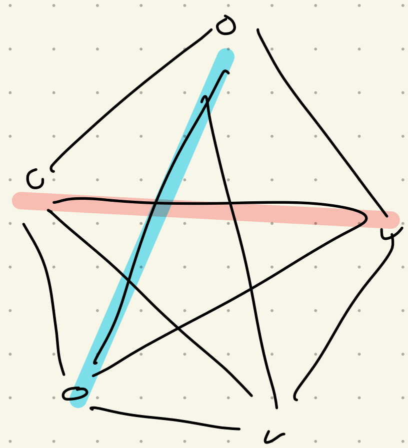
(Implicit construction)

Thm (Erdős) $R(k,k) > n = (1+o(1)) \frac{k}{\sqrt{2}e} \cdot 2^{k/2}$

Pf: Consider a random edge-coloring of K_n

$\forall$ edge $ij \in \binom{[n]}{2}$

all pairs in $[n]$



Color $ij \begin{cases} \text{red} & \text{w./ prob } 1/2 \\ \text{blue} & \text{—— '' ——} \end{cases}$

(independent of all other edges.

$\mathbb{E}$ # monochromatic K_k

linearity of Expectation

$= \binom{n}{k} \cdot Pr($a fixed k-set is monochromatic$)$

$= \binom{n}{k} \left(\frac{1}{2}\right)^{\binom{k}{2}} \cdot 2 < 1$

$\Rightarrow \exists$ a 'good' coloring