Spring 2024 Extremal Combinatorics

Homework 2

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Due April 11

Please submit **5 out of 6 problems** for grading.

1. Let $G = (V, E)$ be a graph with $|V| = n$ and $|E| = \frac{kn}{2}$, where k is a positive integer. Show that G contains an independent set of size at least $\frac{n}{k+1}$.

2. Let $G = (V, E)$ be a graph. For $Y \subseteq V$ and $v \in Y$, define $d_Y(v) = |Y \cap N(v)|$,

$$S(Y) = \sum_{v \in Y} \frac{1}{d_Y(v) + 1}.$$

Suppose $Y = Y_0$ maximizes $S(Y)$ and $|Y_0|$ is smallest possible. Prove that Y_0 is an independent set.

3. Let $G = (V, E)$ be a graph on n vertices. For any edge $e \in E$, denote by $k(e)$ the size of the largest clique in G containing e. Prove that

$$\sum_{e \in E} \frac{k(e)}{k(e) - 1} \leq \frac{n^2}{2}.$$

4. Let a, b be positive real numbers. Place 100 points on the Euclidean plane, where two points possibly share the same position. As a and b vary, what is the largest possible number of (unordered) pairs of points that are either a or b apart?

5. Find the smallest integer n such that one can remove at most n edges from every 1000-vertex 4-regular graph to make it bipartite.

6. Let $n = 2k$ with $k \in \mathbb{N}_+$. Suppose $\mathcal{F}$ is a family of subsets on $[n]$ such that

- for any $A \subseteq [n]$, there exist $B, C \in \mathcal{F}$ such that $B \cup C = A$ and $B \cap C = \varnothing$.

Show that $|\mathcal{F}| \geq (1 - o(1)) \cdot 2^{k+1}$.

Hint: Consider the auxiliary graph on vertex set $\mathcal{F}$, where we put an edge between A and B if and only if $A \cap B = \varnothing$.