# Spring 2024 Extremal Combinatorics

## Homework 1

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Due March 28

Please submit **6 out of 8 problems** for grading. You have to choose **at least two problems** from Problem No.6, 7 and 8.

**1.** Given $n$ irrational numbers $x_1, \dots, x_n$, what is the maximum number of pairs $\{x_i, x_j\}$ such that $x_i + x_j$ is rational? Give an example reaching the maximum.

**2.** Let $G$ be a graph on $n$ vertices and $m$ edges. Show that $G$ has a subgraph $H$ of minimum degree $\delta(H) > \frac{m}{n}$.

**3.** Let $G$ be a triangle-free graph on $n$ vertices. We denote by $e(G)$ and $\delta(G)$ its number of edges and its minimum degree, respectively. Recall the following folklore result:

*$G$ is bipartite if and only if $G$ contains no odd cycle.*

(1) Suppose $G$ is non-bipartite. Show that $e(G) \leq \frac{(n-1)^2}{4} + 1$.

(2) Suppose $\delta(G) > \frac{2n}{5}$. Show that $G$ is bipartite.

**4.** A *V-shape* sequence is a real sequence $x_1, \dots, x_n$ such that

- there exists $i \in \{1, \dots, n\}$ such that $x_1, \dots, x_i$ is decreasing and $x_i, \dots, x_n$ is monotone.

For any fixed positive integer $t$, find the smallest positive integer $n = n(t)$ such that

- every permutation sequence of $1, \dots, n$ contains a V-shape subsequence of length $t$.

**5.** Suppose $A_1, \dots, A_{n+1}$ are non-empty subsets of $\{1, \dots, n\}$. Show that there exist disjoint non-empty index sets $I, J \subseteq \{1, \dots, n+1\}$ such that $\bigcup_{i \in I} A_i = \bigcup_{j \in J} A_j$.

**6.** Let $G$ be a graph on $m$ edges with the following property:

- Any red-blue edge-coloring of $G$ generates either a red $K_s$ or a blue $K_t$.

Show that $m \geq \binom{R(s,t)}{2}$, where $R(s,t)$ denotes the standard 2-color Ramsey number.

**7.** Let $P_5$ be the path on 5 vertices. Determine the extremal number $\mathrm{ex}(100, P_5)$.

**8.** For graphs $G$ and $H$, we say that $G$ is *$H$-saturated* if $G$ does not contain $H$ as a subgraph yet adding any edge into $G$ creates a copy of $H$. The *saturation number* $s_n(H)$ denotes the smallest possible number of edges in an $n$-vertex $H$-saturated graph $G$.

(1) Let $(A_1, B_1), \dots, (A_m, B_m)$ be set pairs with $A_i \cap B_j = \varnothing$ if and only if $i = j$. Show that

$$\sum_{i=1}^{m} \frac{1}{\binom{|A_i|+|B_i|}{|A_i|}} \leq 1.$$

(2) Determine the saturation number of the $r$-clique $s_n(K_r)$.