Hypergraph Lagrangians

Def: Given a k-graph H with vertex set $[n]$ we set $p_H(t) = \sum_{e \in E(H)} \prod_{i \in e} t_i$ for any vector $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$.

Rmk: Note that $p_H(t)$ is the number of edges of the hypergraph obtained from H by replacing each vertex i by t_i copies (for t being a vector of non-negative integers).

Def: The Lagrangian of a k-graph H is

$$\lambda(H) = \max_{\substack{(x_1, \ldots, x_n) \\ x_i \geq 0 \\ \sum x_i = 1}} p_H(x)$$

Def: A k-graph H is dense if $\lambda(H') < \lambda(H)$ for all $H' \subsetneq H$.

Prop: For a k-graph F, we have

$$\pi(F) = \sup\{\lambda(H) \cdot k! : H \text{ is } F\text{-hom-free and dense}\}$$

proof: By definition it is enough to show $\pi(F) = \sup_{H \text{ is F-hom-free}} k!\lambda(H)$

"$\geq$": Let H be an F-hom-free k-graph on n vertices and let $t=(t_1,\dots,t_n)$ be a vector of non-negative integers. Then $H(t)$ is an hom-F-free k-graph on $|t| = \sum t_i$ vertices with density $\frac{p_H(t)}{\binom{|t|}{k}}$. Further, for an integer $m \geq 1$, $H(mt)$ has density $\frac{m^k |t|^k}{\binom{m|t|}{k}} p_H\left(\frac{t}{|t|}\right)$, whence $\pi(F) \geq \lim_{m \to \infty} d(H(mt)) = k!\, p_H\left(\frac{t}{|t|}\right)$.

Since for each vector $x = (x_1, \dots, x_n)$ with $x_i \geq 0$ for all $i \in [n]$ and for each $\varepsilon > 0$, there is some vector $t = (t_1, \dots, t_n)$ of non-negative integers with $p_H\left(\frac{t}{|t|}\right) \geq p_H(x) - \varepsilon$, we get $\pi(F) \geq k!\, \lambda(H)$.

"$\leq$" For every k-graph H on n vertices, we have
$$k!\,\lambda(H) \geq k!\, p_H\left(\tfrac{1}{n}, \dots, \tfrac{1}{n}\right) = \frac{k!}{n^k} e(H) = d(H) - O\left(\tfrac{1}{n}\right).$$
Thus, for every $\varepsilon > 0$, we have
$$\pi(F) = \lim_{n\to\infty} \frac{ex_{hom}(n,F)}{\binom{n}{k}} = \limsup_{n\to\infty} \{ d(H) : H \text{ is } F\text{-hom-free on } n \text{ vertices} \}$$
$$\leq \limsup_{n\to\infty} \{ k!\,\lambda(H) + O\left(\tfrac{1}{n}\right) : H \text{ is } F\text{-hom-free on } n \text{ vertices} \}$$
$$\leq \sup_{H \text{ is } F\text{-hom-free}} k!\,\lambda(H) + \varepsilon$$
and so the result follows.

<u>Obs</u>: If H is a dense k-graph, every pair of vertices in H is contained in an edge.

proof: If two vertices are not contained in an edge together, say i and j, any potential maximiser $x = (x_1, \dots, x_n)$ of $p_H(\cdot)$ could be replaced by x' with $x'_\ell = x_\ell$ for all $\ell \notin \{i, j\}$ and $x_i' = x_i + x_j$ and $x_j' = 0$ or $x_i' = 0$ and $x_j' = x_i + x_j$.

Rmk: This gives an alternative proof for the Erdős-Stone-Simonovits theorem, since only complete graphs are dense and since for some graph F, K_t is F-hom-free if and only if $t < \chi(F)$, we get by the above proposition that

$$\pi(F) = \sup_{\substack{H \text{ is } F\text{-hom-free} \\ \text{and dense}}} 2\lambda(H) = \max_{t < \chi(F)} 2\lambda(K_t^{(2)})$$

To determine $\lambda(K_t^{(2)})$ let $y_1, \dots, y_t$ be such that $\lambda_i \geq 0$ for all i, $\sum y_i = 1$ and

$$\lambda(K_t^{(2)}) = p_{K_t^{(2)}}(y_1, \dots, y_t) = \sum_{ij \in [t]^{(2)}} y_i y_j .$$

By convexity, we have

$$1 = \Big(\sum_{i \in [t]} y_i\Big)^2 \leq t \sum_{i \in [t]} y_i^2 \quad \Rightarrow \quad \sum_{i \in [t]} y_i^2 \geq \frac{1}{t}$$

Using this, we get

$$1 = \Big(\sum y_i\Big)^2 = \sum y_i^2 + 2 \sum_{ij \in [t]^{(2)}} y_i y_j \geq \frac{1}{t} + 2\lambda(K_t^{(2)})$$

whence $\lambda(K_t^{(2)}) \leq \dfrac{t-1}{2t} = \binom{t}{2} \cdot \dfrac{1}{t^2} = p_{K_t^{(2)}}\left(\frac{1}{t}, \dots, \frac{1}{t}\right)$

implying $\lambda_t(K_t^{(2)}) = \dfrac{t-1}{2t}$ and

thereby $\pi(F) = \dfrac{\chi(F) - 2}{\chi(F) - 1}$.

Lemma: Let H be a k-graph on n vertices. Call vertices i and j equivalent if (ij) is an automorphism on H. Then this is indeed an equivalence relation and there are $y_1, \dots, y_n \geq 0$ such that $\sum y_i = 1$, $\lambda(H) = p_H(y_1, \dots, y_n)$ with $y_i = y_j$ whenever i and j are equivalent.

proof: It is easy to check that this is an equivalence relation. For the second part, let $y_1, \dots, y_n \geq 0$ be such that $\sum y_i = 1$, $\lambda(H) = p_H(y_1, \dots, y_n)$ and such that subject to this $\sum y_i^2$ is minimal. Let k, ℓ be equivalent vertices. We claim that $y_k = y_\ell$. Note that

$$p_H\left(\frac{y_1+y_2}{2}, \frac{y_1+y_2}{2}, y_3, \dots, y_n\right) - p_H(y_1, y_2, \dots, y_n)$$

$$= \sum_{\{1,2\} \subseteq e \in E(H)} \left(\left(\frac{y_1+y_2}{2}\right)^2 - y_1 y_2 \right) \prod_{i \in e \setminus \{1,2\}} y_i \geq 0$$

since $\left(\frac{y_1+y_2}{2}\right)^2 - y_1 y_2 = \left(\frac{y_1-y_2}{2}\right)^2 \geq 0$

wlog assume $k=1, \ell=2$

Thus, $\left(\frac{y_1+y_2}{2}, \frac{y_1+y_2}{2}, y_3, \dots, y_n\right)$ was among the candidates for $(y_1, \dots, y_n)$, so $y_1^2 + y_2^2 \leq 2\left(\frac{y_1+y_2}{2}\right)^2$ implying that $y_1 = y_2$.

Cor: $\lambda(K_t^{(k)}) = \frac{1}{t^k}\binom{t}{k}$

Def: Let $H_t^{(k)}$ be a k-graph with vertex set
$$V(H_t^{(k)}) = \{x_i : i \in [t]\} \cup \{y_{ij}^{\ell} : ij \in [t]^{(2)}, \ell \in [k-2]\}$$
and edge set
$$E(H_t^{(k)}) = \{x_i x_j y_{ij}^1 \dots y_{ij}^{k-2} : ij \in [t]^{(2)}\}$$

Thm: (Mubayi): $\pi(H_{t+1}^{(k)}) = \frac{k!}{t^k}\binom{t}{k}$

proof: Note that $K_t^{(k)}(s)$ is $H_{t+1}^{(k)}$-hom-free since every homomorphism φ would map two $x_i\, x_j$ to the same partition class, meaning that $\varphi(x_i)$ and $\varphi(x_j)$ do not lie in an edge together, whereas x_i and x_j are contained together in an edge of $H_{t+1}^{(k)}$. Further note that any dense k-graph H on at least $t+1$ vertices is not $H_{t+1}^{(k)}$-hom-free. To see this, recall that in such an H is contained every pair in an edge. Thus, we can map $x_1, \dots, x_{t+1}$ arbitrarily to $t+1$ distinct vertices $a_1, \dots, a_{t+1}$ and $y_{ij}^1, \dots, y_{ij}^{k-2}$ to some $(k-2)$-set of vertices that together with $a_i a_j$ forms an edge. This tells us
$$\pi(H_{t+1}^{(k)}) = \sup_{\substack{H \text{ is } H_{t+1}^{(k)}\text{-hom-free} \\ \text{and dense}}} k!\,\lambda(H) = \max_{\substack{H \text{ is k-graph} \\ \text{on } s \le t \text{ vertices} \\ \text{and dense}}} k!\,\lambda(H) = k!\,\lambda(K_t^{(k)}) = \frac{k!}{t^k}\binom{t}{k}$$