Supersaturation

Thm: Given a k-graph F and $\varepsilon > 0$, there are $n_0 \in \mathbb{N}$, $\xi > 0$ such that every k-graph H on n vertices with $e(H) \geq (\pi(F)+\varepsilon)\binom{n}{k}$ contains at least $\xi \binom{n}{v(F)}$ copies of F.

proof: Let n_0 be such that $\frac{ex(n_0, F)}{\binom{n_0}{k}} \leq \pi(F) + \frac{\varepsilon}{2}$.

Let H be a k-graph on n vertices with $e(H) \geq (\pi(F)+\varepsilon)\binom{n}{k}$. Given a set $X \subseteq V(H)$, let $e(X)$ be the number of edges of H which are contained in X. Call $X \subseteq V(H)$ good if $e(X) \geq (\pi(F)+\frac{\varepsilon}{2})\binom{n_0}{k}$ and $|X| = n_0$. By double counting

$$(\pi(F)+\varepsilon)\binom{n}{k}\binom{n-k}{n_0-k} \leq \binom{n-k}{n_0-k} e(H) = \sum_{X \in V(H)^{(n_0)}} e(X)$$

$$\leq \binom{n_0}{k} |\{X \subseteq V(H) : X \text{ good}\}| + (\pi(F)+\tfrac{\varepsilon}{2})\binom{n_0}{k} |\{X \in V(H)^{(n_0)} : X \text{ not good}\}|$$

$$\leq \binom{n_0}{k} |\{X \subseteq V(H) : X \text{ good}\}| + (\pi(F)+\tfrac{\varepsilon}{2})\binom{n_0}{k}\binom{n}{n_0}.$$

This implies $\frac{\varepsilon}{2}\binom{n}{n_0} \leq |\{X \subseteq V(H) \text{ good}\}|$

By definition, for each good $X \subseteq V(H)$, $H[X]$ contains a copy of F. By double counting, we obtain

$$\#\text{copies of } F \text{ in } H \cdot \binom{n-v(F)}{n_0-v(F)} = \sum_{X \in V(H)^{(n_0)}} \#\text{copies of } F \text{ in } H[X]$$

$$\geq |\{X \subseteq H \text{ good}\}| \geq \frac{\varepsilon}{2}\binom{n}{n_0}$$

Hence, the number of copies of F in H is at least

$$\frac{\varepsilon}{2} \frac{1}{\binom{n_0}{v(F)}} \binom{n}{v(F)}$$

Def: Given a k-graph and an integer $t \geq 1$, the t-blow-up of F, denoted by $F(t)$, is defined as follows. Let $\{V_x\}_{x \in V(F)}$ be a collection of pairwise disjoint sets of vertices with $|V_x| = t$ for all $x \in V(F)$. Then $V(F(t)) = \bigcup_{x \in V(F)} V_x$ and

$$E(F(t)) = \{ y_{x_1} \ldots y_{x_k} : x_1 \ldots x_k \in E(F) \text{ and } y_{x_i} \in V_{x_i} \text{ for all } i \in [k] \}.$$

Cor: For every k-graph F and integer $t \geq 1$, we have

$$\pi(F) = \pi(F(t))$$

proof: Given $\varepsilon > 0$, a k-graph F and t, let

$$\varepsilon, t^{-1}, v(F)^{-1} \gg \xi, n_0^{-1}, T^{-1} \gg n_1^{-1}$$

Now let H be a k-graph on $n \geq n_1$ vertices with $e(H) \geq (\pi(F) + \varepsilon)\binom{n}{k}$. By the above theorem, H contains at least $\xi \binom{n}{v(F)}$ copies of F. Let $\tilde{H}$ be the auxiliary $v(F)$-graph on $V(H)$ whose edges are the vertex sets on which H has a copy of F. Then $e(\tilde{H}) \geq \xi \binom{n}{v(F)}$. Erdős's earlier theorem thus implies that $\tilde{H}$ contains a copy of $K^{v(F)}_{T,\ldots,T}$. Each copy of F in H corresponding to one of the edges of this $K^{v(F)}_{T,\ldots,T}$ has one of at most $v(F)!$ orientations with respect to the vertex classes of this $K^{v(F)}_{T,\ldots,T}$. By a Ramsey argument, there is a $K^{v(F)}_{t,\ldots,t}$ whose edges all have the same orientation.

Rmk: An easier version of this proof also gives a proof for the Erdős-Stone-Simonovits theorem. Given k-graphs F and H, we say that H is F-hom-free if there is no homomorphism from F to H. Then $ex_{hom}(n,F)$ and $\pi_{hom}(F)$ can be defined analogously to $ex(n,F)$ and $\pi(F)$. The previous argument gives $\pi(F)=\pi_{hom}(F)$.

Def: For a family $\mathcal{F}$ of k-graphs, $ex(n, \mathcal{F})$ and $\pi(\mathcal{F})$ can be defined analogously to $ex(n, F)$ and $\pi(F)$ for a single k-graph F.

Def: The Bollobás 3-graph is

$$B = ([5], \{123, 124, 345\})$$

Thm (Bollobás): $\pi(B, K_4^{(3)-}) = \frac{2}{9}$

Rmk: By supersaturation it follows that $\pi(B) = \frac{2}{9}$

proof: The lower bound is given by the balanced complete 3-partite 3-graph on n vertices, denote its number by $b(n)$.

In the following we will ignore rounding issues.

By induction on n, we will show that if a k-graph H does neither contain B nor $K_4^{(3)-}$, then $e(H) \leq b(n)$. The induction start can easily be checked. If any 3 vertices are incident to at most $b(n) - b(n-3)$ edges, we delete these vertices and are left with a 3-graph on $n-3$ vertices with $e(H) - b(n) + b(n-3)$ edges. If $e(H) > b(n)$, then by induction, this 3-graph, and thereby H, contains a copy of B or of $K_4^{(3)-}$. Thus, we can assume that every triple of vertices is incident to more than $b(n) - b(n-3) = ex(n, K_4^{(2)}) - n + 1$ edges. Because H does not contain a $K_4^{(3)-}$, when we consider any edge xyz, every vertex outside xyz can form an edge with at most one of the pairs in xyz.

Now consider the graph L on $V(H) \setminus \{x,y,z\}$ formed by the links of $x, y,$ and z. Note that since B and $K_4^{(3)-}$ are not contained in H, the link graphs of any two vertices which lie in an edge of H together, must be disjoint. Combining all the above gives

$$e(L) \geq ex(n, K_4^{(2)}) - n + 2 - (n-3) - 1$$
$$= ex(n-3, K_4^{(2)}) + 1$$

Thus, there is some $K_4^{(2)}$ in L, say on a,b,c,d. Note that if any triangle in L would have two edges from the same link, H contains B or $K_4^{(3)-}$. This implies that each matching of size two on a,b,c,d belongs to exactly one of H_x, H_y, H_z. Therefore, each pair of a,b,c,d,x,y,z lies in an edge. As argued above, this means that the links of these vertices are pairwise disjoint. Otherwise, H would contain a $K_4^{(3)-}$ or a B.

Note that for any triple $\alpha\beta\gamma$ of vertices, $e(H_\alpha) + e(H_\beta) + e(H_\gamma)$ is at least the number of edges that $\alpha\beta\gamma$ is incident to, i.e. at least $ex(n, K_4^{(2)}) - n + 2$.

Thus,

$$\binom{6}{2} \sum_{\alpha \in \{a,b,c,d,x,y,z\}} e(H_\alpha) = \sum_{\alpha\beta\gamma \in \{a,b,c,d,x,y,z\}^{(3)}} e(H_\alpha) + e(H_\beta) + e(H_\gamma)$$

$$\geq \binom{7}{3} \left(ex(n, K_4^{(2)}) - n + 2 \right)$$

and so $\sum_{\alpha \in \{a,-,z\}} e(H_\alpha) \geq \frac{7}{3} \left(ex(n, K_4^{(2)}) - n + 2 \right) > \binom{n}{2}$, contradicting the fact, that these links are pairwise disjoint.

Obs: For all $\delta, \varepsilon > 0$ and $m_0 \geq k \geq 2$, there is some n_0 such that every k-graph H on $n \geq n_0$ vertices with at least $(\delta + 2\varepsilon)\binom{n}{k}$ edges contains a k-graph on $m \geq m_0$ vertices with minimum degree at least $(\delta + \varepsilon)\binom{m-1}{k-1}$.

proofsketch: Iteratively delete vertices of too small degree. Because n is much larger than m_0, this process must stop with a k-graph as desired.

Def: $F_{3,3}$ is the hypergraph with $V(F_{3,3}) = \{a,b,c,x,y,z\}$ and $E(F_{3,3}) = \{abc\} \cup \{e \in V(F_{3,3})^{(3)} : |e \cap abc| = 1\}$

Thm (Mubayi, Rödl) $\pi(F_{3,3}) = \frac{3}{4}$

proof: For the lower bound, consider the 3-graph whose vertex set consists of disjoint sets A and B of size $\lfloor \frac{n}{2} \rfloor$ and $\lceil \frac{n}{2} \rceil$, respectively, and whose edges are all triples of vertices that intersect both A and B.

By the previous observation, it is enough to show that for every $\varepsilon > 0$, there is some n_0 such that every 3-graph H on $n \geq n_0$ vertices with minimum degree at least $(\frac{3}{4} + \varepsilon)\binom{n}{2}$ contains $F_{3,3}$.

Let such an H be given. Recall that

$$\pi(K_4^{(3)}) \le \frac{ex(4, K_4^{(3)})}{\binom{4}{3}} = \frac{3}{4}$$

and so we have some vertices $a,b,c,d \in V(H)$ which span a $K_4^{(3)}$ in H.

Consider the following weight function on $V' := V \setminus \{a,b,c,d\}$

$w: V'^{(2)} \to \{0,1,2,3,4\}$ $\quad w(xy) = |\{\alpha \in \{a,b,c,d\} : xy \in E(H_\alpha)\}|$.

Note that we are done if there are $x,y,z \in V'$ such that $w(xyz) = \sum_{e \in \{x,y,z\}^{(2)}} w(e) \ge 11$.

Further, note that the minimum degree means $\quad w(V') \ge 4\left[\left(\frac{3}{4}+\varepsilon\right)\binom{n}{2} - 3\right] \ge \left(3 + \frac{\varepsilon}{2}\right)\binom{|V'|}{2}$

Thus, we are finished by the following lemma.

<u>Lemma:</u> (Bondy, Tuza) Let V be an n-sized set and $w: V^{(2)} \to \{0,1,2,3,4\}$ be a weight function with $w(T) = \sum_{e \in T^{(2)}} w(e) \le 10$ for all $T \in V^{(3)}$.

Then $\quad w(V) \le 2\binom{n}{2} + 2\left\lfloor \frac{n^2}{4} \right\rfloor$.

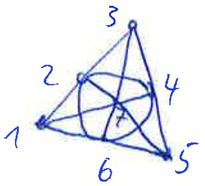
<u>Rmk:</u> This is best possible: Take a balanced bipartition of the vertex set and place weights 2 for the crossing pairs, 4 for the pairs inside any of the partition classes.

proof of Lemma: We perform an induction on n. The induction start can be checked by hand. For the induction step, first assume that $\omega(xy) \le 3$ for all $xy \in V^{(2)}$. In this case we get $\omega(V) \le 3\binom{n}{2} \le 2\binom{n}{2} + 2\lfloor\frac{n^2}{4}\rfloor$. Hence, we may assume that there is some $xy \in V^{(2)}$ with $\omega(xy) = 4$. By the condition $\omega(T) \le 10$ for all $T \in V^{(3)}$, we have $\omega(xz) + \omega(yz) \le 6$ for all $z \in V \setminus \{x,y\}$.
Then we get

$$\omega(T) \le 4 + 6(n-2) + \omega(V \setminus \{xy\})$$

$$\overset{\text{Ind}}{\le} 4 + 6(n-2) + 2\binom{n-2}{2} + 2\left\lfloor\frac{(n-2)^2}{4}\right\rfloor \le 2\binom{n}{2} + 2\left\lfloor\frac{n^2}{4}\right\rfloor$$

<u>Def</u>: The Fano plane is the 3-graph $\mathbb{F}$ with $V(\mathbb{F}) = [7]$ and $E(\mathbb{F}) = \{123, 147, 156, 257, 246, 345, 367\}$



<u>Thm</u> (de Caen, Füredi): $\pi(\mathbb{F}) = \frac{3}{4}$

proof: As before, it is enough to consider a 3-graph H on n vertices, where n is large, with minimum degree $(\frac{3}{4}+\varepsilon)\binom{n}{2}$ and show that it contains $\mathbb{F}$.
As before, we know that there are some a,b,c,d which span a $K_4^{(3)}$ in H.

If for any three of a,b,c,d we see a K_4 in the union of their links such that from each link we use a matching of size two, we are done.

As before, we can consider $\tilde{V} = V \setminus \{a,b,c,d\}$ and $\omega: \tilde{V}^{(2)} \longrightarrow \{0,1,2,3,4\}$ $\omega(e) = |\{\alpha \in \{a,b,c,d\} : e \in E(H_\alpha)\}|$.
The minimum degree condition translates to

$$\omega(\tilde{V}) \geq 4\left(\left(\tfrac{3}{4}+\varepsilon\right)\binom{n}{2} - 3\right) \geq \left(3+\tfrac{\varepsilon}{2}\right)\binom{n}{2}$$

Due to a result by Füredi and Kündgen that we will not prove here, this implies that there are $x_1, \dots, x_4 \in \tilde{V}$ with $\omega(\{x_1,\dots,x_4\}) \geq 21$.
Now consider the following bipartite graph $\mathcal{G}$. Let one partition class, A, consist of a,b,c,d and the other, B, of all the maximal matchings on $x_1, \dots, x_4$. Place an edge between $\alpha \in \{a,b,c,d\}$ and $\{e,f\} \in B$ if $e,f \in H_\alpha$.
Since $\omega(\{x_1,\dots,x_4\}) \geq 21$, there are at most three ~~edges~~ pairs missing in $\mathcal{G}$. It can easily be seen that this means that $\mathcal{G}$ contains a matching of B. This corresponds to a copy of $\mathbb{F}_n$ in H.