Lecture 1

Q: Extremal Combinatorics: What is the threshold in some parameter ϱ of a discrete structure $\mathcal{D}$ above which $\mathcal{D}$ is guaranteed to satisfy property $\mathcal{P}$?

Ex: • ϱ: min degree $\mathcal{D}$: graph $\mathcal{P}$: containing a Hamiltonian cycle

→ Dirac's theorem

• ϱ: #sets $\mathcal{D}$: family $\mathcal{F} \subseteq \binom{[n]}{k}$ $\mathcal{P}$: having two sets which do not intersect

→ Erdős Ko Rado theorem

• ϱ: edge density $\mathcal{D}$: graph with many vertices $\mathcal{P}$: containing a clique K_t

→ Turán's theorem

Def: A hypergraph $H=(V,E)$ consists of a vertex set V and an edge set $E \subseteq \mathcal{P}(V)$. H is k-uniform if $|e|=k$ for all $e \in E$. Sometimes we call a k-uniform hypergraph simply a k-graph.

Ex: A graph is a 2-graph.

Def: Given a k-graph F and $n \in \mathbb{N}$, the extremal number of n and F is the maximum number of edges in an F-free k-graph on n vertices, we denote it by $ex(n,F)$. The Turán density of F is $\pi(F) = \lim_{n\to\infty} \frac{ex(n,F)}{\binom{n}{k}}$.

<u>Obs</u>: The Turán density is well-defined, i.e. for every k-graph F, $\lim_{n\to\infty} \frac{ex(n,F)}{\binom{n}{k}}$ exists.

proof: We show that $\frac{ex(n,F)}{\binom{n}{k}}$ is non-increasing.

Since it is bounded from below by 0, the observation will follow. Let H be a k-graph on $n+1$ vertices with $e(H) > \frac{ex(n,F)}{\binom{n}{k}}\binom{n+1}{k}$. We need to argue that $F \subseteq H$. For this, we will find a vertex $v \in V(H)$ such that $e(H\setminus v) > ex(n,F)$. Note that double counting gives

$$\sum_{x\in V(H)} e(H\setminus x) = (n+1-k)\, e(H) > \frac{(n+1-k)\, ex(n,F)}{\binom{n}{k}}\binom{n+1}{k}$$

and so for some $v \in V(H)$, we indeed have $e(H\setminus v) > ex(n,F)$.

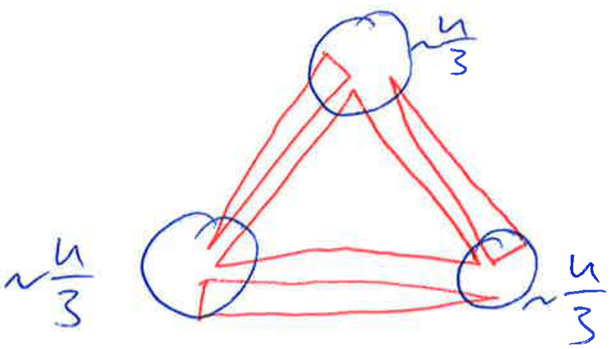
<u>Rmk</u>: For graphs, i.e. $k=2$, the Erdős–Stone–Simonovits theorem determines the Turán density of any graph F to be $\pi(F) = \frac{\chi(F)-2}{\chi(F)-1}$.

<u>Pbm</u> (Turán, 1941): Determine $\pi(K_4^{(3)})$

<u>Ex</u>: Let H be the 3-graph with vertex set $V = A \cup B \cup C$ with $|A| \approx |B| \approx |C| \approx \frac{n}{3}$ and edge set

~~$E = \{abc : a\in A, b\in B, c\in C\} \cup \{a_1a_2b : a_i \in A, b\in B\} \cup \{b_1b_2c : b_i\in B, c\in C\} \cup \{$~~

$$E = \{abc : a \in A, b \in B, c \in C\} \cup \{a_1 a_2 b : a_1, a_2 \in A, b \in B\}$$
$$\cup \{b_1 b_2 c : b_1, b_2 \in B, c \in C\} \cup \{c_1 c_2 a : a \in A, c_1, c_2 \in C\}$$



It can easily be seen that H is $K_4^{(3)}$-free and ~~a~~ asymptotically has density $\frac{5}{9}$.

Thus $\pi(K_4^{(3)}) \geq \frac{5}{9}$

Obs: $ex(n, K_4^{(3)}) \le \frac{2n-3}{9}\binom{n}{2}$ and so $\pi(K_4^{(3)}) \le \frac{2}{3}$

proof: Let $H=(V,E)$ be a $K_4^{(3)}$-free 3-graph with $|V|=n$.
For distinct $x,y\in V$, let $d(x,y)=|\{z\in V: xyz\in E\}|$ be the pair degree of x and y. Note that if $xyz\in E$, H being $K_4^{(3)}$-free implies

$$d(x,y)+d(x,z)+d(y,z)\le 2(n-3)+3 = 2n-3.$$

Hence, we get by double counting that

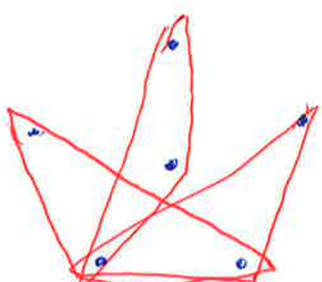
$$\sum_{xy\in V^{(2)}} d(x,y)^2 = \sum_{xyz\in E} d(x,y)+d(x,z)+d(y,z) \le (2n-3)|E|.$$

Convexity implies

$$\sum_{xy\in V^{(2)}} d(x,y)^2 \ge \binom{n}{2}\left(\frac{\sum d(x,y)}{\binom{n}{2}}\right)^2 = \frac{9|E|^2}{\binom{n}{2}}.$$

Combining these bounds yields $|E|\le \frac{2n-3}{9}\binom{n}{2}$

Ex (Frankl, Füredi): Let H_1 be the 3-graph with vertex set $[6]$ and edge set
$\{123, 234, 345, 451, 512\} \cup \{146, 256, 316, 426, 536\}$



plus rotations

Then define H_n iteratively by taking 6 copies of H_{n-1}: $H_{n-1}^{(1)}, H_{n-1}^{(2)} \dots H_{n-1}^{(6)}$ and adding all edges of the form $x_i x_j x_k$ with $x_i\in V(H_{n-1}^{(i)})$, $x_j\in V(H_{n-1}^{(j)})$, $x_k\in V(H_{n-1}^{(k)})$ and $ijk\in E(H_1)$.

It can easily be checked that asymptotically H_n has density $\frac{2}{7}$ and is $K_4^{(3)-}$-free.
Thus $\pi(K_4^{(3)-}) \geq \frac{2}{7}$ and this is conjectured to be optimal.

Obs: (de Caen) $\pi(K_4^{(3)-}) \leq \frac{1}{3}$

proof: Let H be a $K_4^{(3)-}$-free 3-graph on n vertices. Let S denote the number of 4-sets of vertices on which H sees two edges. Since H has at most two edges on every 4-set of vertices, we have $S = \sum_{xy \in V(H)^{(2)}} \binom{d(xy)}{2}$. By convexity, we get setting $e(H) = \alpha \binom{n}{3}$,

$$S = \sum_{xy \in V(H)^{(2)}} \binom{d(xy)}{2} \geq \binom{n}{2} \binom{\frac{\sum d(xy)}{\binom{n}{2}}}{2} = \binom{n}{2} \binom{\frac{3|E(H)|}{\binom{n}{2}}}{2} = \binom{n}{2} \binom{\frac{3\alpha\binom{n}{3}}{\binom{n}{2}}}{2} = \frac{\alpha^2 n^4}{4} + O(n^3).$$

Double counting the pairs (T, e) of a 4-set of vertices with (exactly) two edges and one of its edges gives

$$2S \leq (n-3)\, e(H) \quad \text{whence} \quad \frac{\alpha^2 n^4}{2} + O(n^3) \leq (n-3)\,\alpha \binom{n}{3}$$

and so $\alpha \leq \frac{1}{3} + O\left(\frac{1}{n}\right)$

Def: A k-graph $H=(V,E)$ is k-partite if there is a partition $V=V_1 \cup \ldots \cup V_k$ such that $|e \cap V_i| = 1$ for all $e \in E$ and $i \in [k]$.

Thm: (Erdős) A k-graph F satisfies $\pi(F)=0$ if and only if F is k-partite.

Cor: There is no k-graph F with $\pi(F) \in (0, \frac{k!}{k^k})$

proof: First consider a k-graph F that is not k-partite. Then F is not contained in the complete balanced k-partite k-graph. This gives $\pi(F) \geq \frac{k!}{k^k}$.

To show that for a k-partite k-graph F we have $\pi(F)=0$, we introduce the following important definitions.

Def: For a k-graph $H=(V,E)$ and a vertex $x \in V$, the link of H at x, denoted by H_x, is the $(k-1)$-graph with $V(H_x)=V(H)$ and $E(H_x)=\{e \in V(H)^{(k-1)} : e \cup \{x\} \in E(H)\}$.

Def: A homomorphism from a k-graph H to a k-graph F is a map $\varphi: V(H) \to V(F)$ such that $\varphi(e) \in E(F)$ for all $e \in E(H)$.

Further, we set $hom(H,F) = \{\varphi: V(H) \to V(F) : \varphi \text{ is hom}\}$

and $t(H,F) = \dfrac{\hom(H,F)}{v(F)^{v(H)}}$

<u>Not</u>: For ~~[illegible]~~ integers $m_1, \ldots, m_k \geq 1$ we write $K^{(k)}_{m_1,\ldots,m_k}$ for the complete k-partite graph with partition classes $V_1, \ldots, V_k$ with $|V_i| = m_i$ for all i.

Now we begin our proof that a k-partite k-graph F satisfies $\pi(F) = 0$. First, we prove the following claim.

<u>Claim</u>: Every k-graph H satisfies

$$t(K^{(k)}_{m_1,\ldots,m_k}, H) \geq t(K^{(k)}_{1,\ldots,1}, H)^{m_1 \cdot \ldots \cdot m_k}$$

proof: We perform an induction on k

$k=1$: $\quad t(K^{(1)}_m, H) = \dfrac{e(H)^m}{v(H)^m} = t(K^{(1)}_1, H)^m$

$k-1 \to k$: Set $v(H) = n$. $V(H) = V$

$$\hom(K^{(k)}_{m_1,\ldots,m_k}, H) = \sum_{v(1),\ldots,v(m_k) \in V(H)} \hom(K^{(k-1)}_{m_1,\ldots,m_{k-1}}, H_{v(1)} \cap \ldots \cap H_{v(m_k)})$$

$$= n^{m_1+\ldots+m_{k-1}} \sum t(K^{(k-1)}_{m_1,\ldots,m_{k-1}}, H_{v(1)} \cap \ldots \cap H_{v(m_k)})$$

$$\overset{IH}{\geq} n^{m_1+\ldots+m_{k-1}} \sum t(K^{(k-1)}_{1,\ldots,1}, H_{v(1)} \cap \ldots \cap H_{v(m_k)})^{m_1 \cdot \ldots \cdot m_{k-1}}$$

$$\overset{\text{convexity}}{\geq} n^{m_1+\ldots+m_k} \left(\sum \frac{t(K^{(k-1)}_{1,\ldots,1}, H_{v(1)} \cap \ldots \cap H_{v(m_k)})}{n^{m_k}} \right)^{m_1 \cdot \ldots \cdot m_{k-1}}$$

To bound $\sum t(K^{(k-1)}_{1,\ldots,1}, H_{v(1)} \cap \ldots \cap H_{v(m_k)})$, we estimate $\sum \hom(K^{(k-1)}_{1,\ldots,1}, H_{v(1)} \cap \ldots \cap H_{v(m_k)})$.

$$\sum_{v(1),\dots,v(m_k)\in V} \hom\left(K^{(k-1)}_{1,\dots,1}, H_{v(1)}\cap\dots\cap H_{v(m_k)}\right)$$

$$= (k-1)! \sum_{e\in V(H)^{(k-1)}} \left|\{(v(1),\dots,v(m_k))\in V(H)^{m_k} \mid \forall i\in[m_k]\ e\in H_{v(i)}\}\right|$$

$$= (k-1)! \sum_{e\in V(H)^{(k-1)}} |\{v\in V \mid e\in H_v\}|^{m_k}$$

$$\overset{\text{convexity}}{\geq} (k-1)! \binom{n}{k-1} \left(\frac{\sum_{e\in V(H)^{(k-1)}} |\{v\in V\mid e\in H_v\}|}{\binom{n}{k-1}} \right)^{m_k}$$

$$= (k-1)! \binom{n}{k-1}^{1-m_k} \left(k\cdot e(H)\right)^{m_k}$$

$$\geq (k-1)! \left(\frac{n^{k-1}}{(k-1)!}\right)^{1-m_k} \left(k\cdot e(H)\right)^{m_k}$$

$$= n^{(k-1)(1-m_k)} \left(k!\, e(H)\right)^{m_k} = n^{k-1+m_k}\, t(\underbrace{e}_{K^{(k)}_{1,\dots,1}}, H)^{m_k}$$

Plugging this in above yields

$$\hom\left(K^{(k)}_{m_1,\dots,m_k}, H\right) \geq n^{m_1+\dots+m_k}\, t(\underbrace{e}_{K^{(k)}_{1,\dots,1}}, H)^{m_1\cdots m_k}.$$

$\boxed{\text{Claim}}$

Now let F be a k-partite k-graph and let $m_1,\dots,m_k \geq 1$ be integers such that $F \subseteq K^{(k)}_{m_1,\dots,m_k}$. Let $\varepsilon > 0$ be arbitrary, pick $n_0 \gg \varepsilon^{-1}$ and let H be a k-graph on $n \geq n_0$ vertices with $e(H) > \varepsilon\binom{n}{k}$. The claim implies

$$t\left(K^{(k)}_{m_1,\dots,m_k}, H\right) \geq t\left(K^{(k)}_{1,\dots,1}, H\right)^{m_1\cdots m_k}$$

$$= \left(\frac{k!\, e(H)}{n^k}\right)^{m_1\cdots m_k} > \left(\frac{\varepsilon}{2}\right)^{m_1\cdots m_k}.$$

This implies that there must be an injective homomorphism from $K^{(k)}_{m_1,\dots,m_k}$ to H and thus $F \subseteq H$