Problems

Proposed by Jozsef Balogh, University of Illinois at Urbana-Champaign

The famous Tree Packing Conjecture (TPC) posed by Gyárfás states:

Conjecture. *Any set of $n-1$ trees $T_n, T_{n-1}, \dots, T_2$ such that T_i has i vertices, packs*[1] *into K_n.*

Bollobás suggested a weakening of TPC in the Handbook of Combinatorics:

Conjecture. *For every $k \geq 1$, there is an $n(k)$ such that if $n \geq n(k)$, then arbitrary set of k trees $T_1, T_2, \dots, T_k$ such that T_i has $n-i+1$ vertices pack into K_n.*

There are several partial results, like it is known for $k \leq 5$ (unpublished). Other results on TPC when there are maximum degree conditions on the trees. I trust that the second conjecture is within the reach, in particular in view of results of Balogh and Palmer, see below:

If we let the complete graph to have one more vertex than allowed by the first conjecture, then we can pack many trees without conditions on their structure.

Theorem. *Let n be sufficiently large and $t = n^{1/4}$. If $T_1, T_2, \dots, T_t$ are trees such that $|T_i| = n-i+1$ for every i, then $T_1, T_2, \dots, T_t$ pack into K_{n+1}.*

Eliminating a single case from the proof of the Theorem gives the following proposition.

Proposition. *Let n be sufficiently large and $t = n^{1/4}$. If $T_1, T_2, \dots, T_t$ are trees such that $|T_i| = n-i+1$ and T_i is not a star for each i, then $T_1, T_2, \dots, T_t$ pack into K_n.*

References

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[1] Packing means that the edge set of K_n is partitioned into classes $E_n, \dots, E_2$, such that E_i is isomorphic to T_i.

Partition problem for clique-free graphs

Proposed by Felix Christian Clemen, Karlsruhe Institute of Technology

Let G be a graph and A a vertex-subset of G. We denote by $e(A)$ the number of edges in G with both vertices from A. Let

$$D_2(G) := \min_{A \subseteq V(G)} (e(A) + e(A^c)).$$

Sudakov [5] proved that every K_4-free n-vertex graph G can be made bipartite by removing at most $n^2/9$ edges, i.e. $D_2(G) \leq n^2/9$. Note that this result is sharp because the balanced complete 3-partite graph requires at least $n^2/9$ edges removed to make it bipartite. Sudakov [5] also conjectured the following generalization.

Conjecture (Sudakov [5])**.** *Fix $r \geq 3$. For every n-vertex K_{r+1}-free graph G, it holds that*

$$D_2(G) \leq \begin{cases} \frac{(r-1)^2}{4r^2} \cdot n^2 & r \text{ odd, and} \\ \frac{r-2}{4r} \cdot n^2 & r \text{ even.} \end{cases}$$

Hu, Lidický, Martins, Norin, and Volec [2] verified Conjecture for $r = 5$ using the method of flag algebras. Recently Reiher [4], building up on work of Liu and Ma [3], proved the corresponding sparse-half-version of Sudakov's result on K_4-free graphs: Every K_4-free graph contains a set of size $n/2$ spanning at most $n^2/18$ edges. If true, the following conjecture would generalize both Sudakov's and Reiher's result.

Conjecture (Balogh, Clemen, Lidický [1])**.** *Let n be an even positive integer and G be a K_4-free graph on n vertices. Then there exists a partition of its vertex set $V(G) = A \cup B$ such that $|A| = |B| = n/2$ and $e(A) + e(B) \leq n^2/9$.*

If true, Conjecture is sharp, because the complete balanced 3-partite graph satisfies $e(A)+e(B) \geq n^2/9$ for any partition $V = A \cup B$ with $|A| = |B| = n/2$.

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A problem on clique cover numbers

Proposed by Jialin He, The Hong Kong University of Science and Technology

A set $\mathcal{C}$ of cliques is a p-clique cover of G if for every p-clique $S \subseteq V(G)$ there is a clique $Q \in \mathcal{C}$ that covers S (i.e., S is a subgraph of Q). Let $cc_p(G)$ denote the p-clique cover number of G, that is the minimum number of cliques in a p-clique cover of G.

When $p = 2$, $cc_2(G)$ also referred to as the intersection number of a graph. Erdős, Goodman and Pósa [2] proved that $cc_2(G) \leq \lfloor \frac{n^2}{4} \rfloor$ for every graph on n vertices. Equality holds if and only if G is isomorphic to $T_2(n)$, here $T_p(n)$ denotes the p-partite Turán graph on n vertices. Dau, Milenkovic and Puleo [1] proposed the following conjecture and proved the case when $p = 3$.

Conjecture. *If $p \geq 3$ and $n \geq p$ are integers, then for every n-vertex graph G, $cc_p(G) \leq cc_p(T_p(n))$. Equality holds if and only if G is isomorphic to $T_p(n)$.*

References

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Minimum size of a k-base for finite sets

Proposed by Dias Mattos, Leticia, University of Illinois Urbana-Champaign

Shotgun assembly of random graphs

For a graph G, let $N_r(G)$ be the graph induced by the vertices at distance at most r from v, where the vertices are unlabelled except for the vertex v. For an integer $r \geq 1$ and graphs G and H, we say G and H have isomorphic r-neighbourhoods if there is a bijection $\varphi : V(G) \to V(H)$ such that for each vertex v of G there is an isomorphism from the r-neighbourhood $N_r(v)$ around v in G to the r-neighbourhood $N_r(\varphi(v))$ around $\varphi(v)$ in H which maps v to $\varphi(v)$. We say that G is reconstructible from its r-neighbourhoods (or r-reconstructible) if every graph with r-neighbourhoods isomorphic to those of G, is in fact isomorphic to G. The general problem is to determine for what range of p a random graph $G \in G(n,p)$ is reconstructible (or non-reconstructible) from its r-neighbourhoods with high probability (i.e. with probability tending to 1 as n tends to infinity).

The following problem was posed by Johnston, Kronenberg, Roberts and Scott [**?**]:

Question. *Determine when $G(n,p)$ is 2-reconstructible. Is there a threshold around $n^{-3/4}$ (up to a polylogarithmic factor)?*

Largest subgraph from a hereditary property in a random graph

Let $\mathcal{P}$ be an arbitrary hereditary property of graphs. We assume that $\mathcal{P}$ is non-trivial, i.e., it contains all edgeless graphs and misses some graph. For a graph G, let $\text{ex}(G, \mathcal{P})$ denote the maximum number of edges of a subgraph of G that belongs to $\mathcal{P}$; the above definition of non-triviality guarantees that this number is well-defined. In [**?**], Alon, Krivelevich and Samotij determined, for every fixed edge probability $p \in (0,1)$, the typical asymptotic value of $\text{ex}(G(n,p), \mathcal{P})$ for the random graph $G(n,p)$ as n tends to infinity. In the concluding remarks section in [**?**], they posed the following questions:

Question. *Let $p \in (0,1)$ be constant. If the hereditary property $\mathcal{P}$ misses a bipartite graph, then $ex(G(n,p), \mathcal{P}) \leq n^{2-\epsilon}$, for some $\epsilon > 0$.*

Question. *Determine $ex(G, \mathcal{P})$ when p tends to 0.*

Random subgraphs of the hypercube

Analogous to the case of the binomial random graph $G(n,p)$, it is known that the behaviour of a random subgraph of an n-dimensional hypercube, where we include each edge independently with probability p, which we denote by Q^n_p, undergoes a phase transition when p is around $1/n$. More precisely, standard arguments show that significantly below this value of p, with probability tending to one as $n \to \infty$ (whp for short) all components of this graph have order $O(n)$, whereas Ajtai, Komlós and Szemerédi showed that significantly above this value, in the supercritical regime, whp there is a unique 'giant' component of order $\Theta(2^n)$. In $G(n,p)$, much more is known about the complex structure of the random graph which emerges in this supercritical regime. For example, it is known that in this regime whp $G(n,p)$ contains paths and cycles of length $\Omega(n)$.

In [**?**], Erde, Kang and Krivelevich obtained an analogous result in Q^n_p. In particular, they showed that if $p = (1+\epsilon)/n$, where $\epsilon > 0$ is some positive constant, then whp Q^n_p contains a cycle of length $\Omega\left(\frac{2^n}{n^3(\log n)^3}\right)$. At the end of their paper, they posed the following questions:

Question. *Let $\epsilon > 0$ and $p = (1+\epsilon)/n$. Is it true that whp Q^n_p contains a cycle of length $\Omega(2^n)$?*

Question. *Let $p = \omega(1/n)$. Is it true that whp Q^n_p contains a path of length $(1 - o(1))2^n$?*

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Minimum size of a k-base for finite sets

Proposed by Haoran Luo, University of Illinois Urbana-Champaign

Let X be a set of size n. A family $\mathcal{H} \subseteq \mathcal{P}(X)$ is a k-base for X if every subset $S \subseteq X$ is the union of at most k sets in $\mathcal{H}$. For example, let $X_1, X_2, \ldots, X_k$ be a partition of X where $||X_i| - |X_j|| \leq 1$; then

$$\mathcal{H}_{n,k} := \bigcup_{i=1}^{k} \mathcal{P}(X_i)$$

is a k-base. In 1993, Erdős (see [3]) proposed the problem of determining the minimum size of a 2-base and made the following conjecture.

Conjecture. *If X is a set of size n and family $\mathcal{H} \subseteq \mathcal{P}(X)$ is a 2-base for X, then*

$$|\mathcal{H}| \geq |\mathcal{H}_{n,2}|.$$

Frein, Lévêque, and Sebő [2] made an analogous conjecture for all k.

Under a stronger assumption that every subset $S \subseteq X$ is the union of at most k *disjoint* sets in $\mathcal{H}$, Ellis and Sudakov [1] confirmed Conjecture for sufficiently large n when $k = 2$, and for sufficiently large n that are multiple of k when $k \geq 3$.

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Problems

Proposed by Sam Mattheus, University of California San Diego

In recent work with Jacques Verstraete [3], there were some questions that we were not (yet?) able to answer. Here are two of them.

Our starting point is the graph $\mathrm{NU}(3, q^2)$, which exists for every prime power q. Its relevant properties are as follows, see for example [1, Section 3.1.6]. Recall that a *strongly regular graph* with parameters (v, k, a, c) is a k-regular graph on n vertices such that every two adjacent (resp. non-adjacent) vertices have a (resp. c) common neighbors.

Theorem. *The graph $H_q = \mathrm{NU}(3, q^2)$ satisfies the following properties.*

1. *It is strongly regular with parameters*
$$n = q^4 - q^3 + q^2, \qquad k = (q+1)(q^2-1), \qquad a = 2q^2 - 2, \qquad c = (q+1)^2;$$
2. *H_q has eigenvalues k, $q^2 - q - 2$ and $-q-1$;*
3. *it is the edge-disjoint union of $q^3 + 1$ maximal cliques each of size q^2. We denote this set of cliques by $\mathcal{C}$;*
4. *every two cliques in $\mathcal{C}$ have exactly one vertex in common;*
5. *every copy of K_4 in H_q has at least three vertices in a clique of $\mathcal{C}$.*

Property 4 has a very interesting consequence: the graph H_q can be made K_4-free by replacing every clique in $\mathcal{C}$ by a bipartite graph. Typically, we will replace a clique by a complete bipartite graph in a random fashion. This preserves the density (up to constant) while keeping an appropriate amount of (pseudo)randomness. This procedure is sometimes called a 'random block construction' in the literature.

The spectrum of a random block construction. For example, David Conlon used this idea to construct a sequence of triangle-free pseudorandom graphs, coming from the collinearity graph of generalized quadrangles [2]. The latter family of strongly regular graphs is defined for every prime power q and has parameters $(q^3+q^2+q+1, q^2+q, q-1, (q+1)^2)$. From the theory of strongly regular graphs, it follows that its non-trivial eigenvalues are $-q-1$ and $q-1$. It would be interesting to know what we can say about the behavior of the smallest eigenvalue by once we perform the random block construction. While in our case, the smallest eigenvalue drops from $-q-1$ to at most $-q^2$ whp, as pointed out to us by Carl Schildkraut, it could be the case that in David's construction the smallest eigenvalue is at least $-cq$ for some $c > 0$. This would lead to a spectral proof of $r(3,t) = \Omega(t^2/\log^2 t)$ using the ideas of [3].

Problem. *Show that whp the smallest eigenvalue of the triangle-free graphs constructed by Conlon is at least $-cq$ for some constant $c > 0$.*

Counting independent sets. In [3] we count the number of independent sets of size $t = q\log^2 q$ using the container method (or rather the Kleitman-Winston algorithm). It turns out that this number is at most $\binom{cq^2}{t} \approx (cq/\log^2 q)^t$ for some constant $c > 0$. Remark that H_q (and hence H_q^*) has independent sets of size roughly q^2, so this count is sharp up to constants. If one would be able to prove a similar statement for slightly smaller independent sets, i.e. $t = q\log q$, then our proof would actually show $r(4,t) = \Theta(t^3/\log^2 t)$. General results on container methods seem to be unable to break through the $\log^2 q$ barrier, but perhaps there is a clever way to adapt the ideas to the more structured graph model H_q^*.

Problem. *Show that the number of independent sets in H_q of size $t = q \log q$ is at most $(cq/\log q)^t$ for some $c > 0$.*

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Number of independent sets in k-connected bipartite graph

Proposed by Eero Räty, Umeå University

Consider the partially ordered set on $[t]^n := \{0, \ldots, t-1\}^n$ equipped with the natural coordinate-wise ordering. Let $A(t,n)$ denote the number of antichains of this poset. The quantity $A(t,n)$ has a number of combinatorial interpretations, see e.g. [4]. A number of results in the literature [5, 6, 7] show that $\log_2 A(t,n) = (1+o(1)) \cdot \alpha(t,n)$, where $\alpha(t,n)$ is the width of $[t]^n$, and the $o(1)$ term goes to 0 for t fixed and n tending to infinity. Recently, with Falgas-Ravry and Tomon [1], we proved that there exists an absolute constant c such that for every $t, n \geq 2$ we have $\log_2 A(t,n) \leq \left(1 + c \cdot \frac{(\log n)^3}{n}\right) \cdot \alpha(t,n)$.

Even less is known regarding the lower bounds. We pointed out that an easy argument implies that the $o(1)$-term must be at least $2^{-c'n}$ for some constant c', which is known to be the case for the Boolean lattice [3].

Let N and n be positive integers with $n|N$. A result of Kahn [2] implies that the number of independent sets in n-regular bipartite graph G on $2N$ vertices is at most $(2^{n+1}-1)^{N/n} = \exp_2\left(\left(1 + \frac{1+o(1)}{n}\right) N\right)$. This is tight, with equality attained by choosing G to be disjoint copies of $K_{n,n}$. However, when compared to bipartite graphs arising naturally from the hypergrid, this example is not particularly interesting as the graph splits into a number of connected components.

What happens if we further impose that G needs to be k-connected for some k? Keeping the hypergrid example in our mind, we are interested up to the regime when $k = \beta n$ where β is some fixed small constant, and where $N \gg n \gg 1$. By considering various modifications[2] of the optimal example consisting of disjoint union of $K_{n,n}$'s, it seems plausible that the error term in the Kahn's bound could be improved under some additional assumption concerning the connectivity of G. This leads to the following question, which might guide us towards obtaining an improved bound for $A(t,n)$.

Question. *Let β be a fixed constant with $\beta < \frac{1}{4}$. Does there exist $\gamma \in (0,1)$ so that whenever G is a n-regular bipartite graph on $2N$ vertices which is k-connected for some $k \leq \beta n$ and N is sufficiently large in terms of k and n, then the number of independent sets in G is at most $\exp_2\left(\left(1 + O\left(\gamma^k\right)\right) N\right)$?*

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[2] A natural way to ensure k-connectivity when k is even is as follows. Write $A_1, \ldots, A_t$ and $B_1, \ldots, B_t$ for the vertex sets of $t = \frac{N}{t}$ $K_{n,n}$'s. Order them in a circle, delete a matching of size $\frac{k}{2}$ between each A_i and B_i while add such a matching between B_i and A_{i+1}. It is not too hard to check that the number of independent sets is at most $2^{N(1+\alpha_{n,k})}$ where one can take $\alpha_{n,k} \approx \frac{(3/4)^{k/2}}{n \log(2)}$

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Problem

Proposed by Chong Shangguan, Shandong University

We will discuss a problem of Erdős, Frankl, and Füredi [1] on cover-free families. A family $\mathcal{F} \subseteq 2^{[n]}$ is said to be d-cover-free if for arbitrary distinct $d+1$ members $A_0, A_1, \ldots, A_d \in \mathcal{F}$, $A_0 \nsubseteq \cup_{i=1}^{d} A_i$. The maximum size of (uniform) cover-free families has been studied extensively, see e.g. [1, 2]. We will focus on another problem with slightly different flavor, as stated below.

Clearly the set of singletons $\mathcal{T} = \{\{1\}, \ldots, \{n\}\}$ is d-cover-free for every $d \in [n]$. However such a cover-free family is trivial in the sense that the number of members in this family is equal to the size of the underlying set. A d-cover-free family $\mathcal{F} \subseteq 2^{[n]}$ is called non-trivial if $|\mathcal{F}| > n$. Given d, let $T(d)$ be the smallest n such that there exists a non-trivial d-cover-free family defined on $[n]$.

A projective plane of order $d+1$ is a point-line structure $(\mathcal{P}, \mathcal{L})$ with $|\mathcal{P}| = |\mathcal{L}| = (d+1)^2 + (d+1) + 1$, with $(d+1)+1$ points on each line, $(d+1)+1$ lines through each point, and every two distinct lines has exactly one intersection. Therefore $\mathcal{L}$ is $(d+1)$-cover-free (however still trivial).

By deleting a line together with all points on this line from $(\mathcal{P}, \mathcal{L})$ one obtains an affine plane of order $d+1$, which is a point-line structure $(\mathcal{P}', \mathcal{L}')$ with $|\mathcal{P}'| = (d+1)^2$, $|\mathcal{L}'| = (d+1)^2 + (d+1)$, with $(d+1)$ points on each line, $(d+1)+1$ lines through each point, and every two distinct lines have at most one intersection. Therefore $\mathcal{L}'$ is d-cover-free, which implies that $T(d) \leq (d+1)^2$, as long as such structure exists. It is known that projective and affine planes exist for every prime power d.

Erdős, Frankl and Füredi conjectured that the above construction is essentially the best possible.

Conjecture ([1])**.** *$\lim_{d\to\infty} \frac{T(d)}{d^2} = 1$, or even stronger, $T(d) \geq (d+1)^2$.*

Currently the best known lower bound is due to Shangguan and Ge [3], who showed that $T(d) \geq \frac{15+\sqrt{33}}{24} d^2$, using the classic result of Erdős and Gallai on the maximum number of edges in a graph with bounded matching number.

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Problem

Proposed by Tuan Tran, University of Science and Technology of China

1 Isolation lemma

Consider a hypergraph $\mathcal{H}$ with vertex set $[n]$. A weight is simply a functions $w\colon [n] \to [M]$. This weighting extends naturally to edges $e \in E(\mathcal{H})$ by letting $w(e) = \sum_{i\in e} w(i)$.

We say that $e \in E(\mathcal{H})$ is a *min-weight edge* with respect to w if for all edges $e' \in E(\mathcal{H})$ we have $w(e') \geq w(e)$. A weight $w \in [M]^n$ is *isolating* if there is exactly one min-weight edge, that is, there is an edge $e \in E(\mathcal{H})$ such that $w(e') > w(e)$ for all $e' \in E(\mathcal{H}) \setminus \{e\}$.

Given any hypergraph $\mathcal{H}$, we define

$$Z(\mathcal{H}, M) = \{w \in [M]^n : w \text{ is isolating with respect to } \mathcal{H}\}.$$

The Isolation Lemma asserts that $|Z(\mathcal{H}, M)| \geq (1 - \frac{n}{M})M^n$. The lemma has important applications in computer science, such as the Valiant–Vazirani theorem and Toda's theorem in computational complexity theory.

Faber and Harris [2] made the following conjecture.

Conjecture (Faber–Harris, 2018)**.** $|Z(\mathcal{H}, M)| \geq n \sum_{k=1}^{M-1} k^{n-1}$.

The bound is attained, for example, by taking $E(\mathcal{H}) = \{\{1\}, \ldots, \{n\}\}$. We remark that Faber and Harris verified their conjecture for graphs.

2 Touching simplices

How many d-dimensional simplices can be positioned in $\mathbb{R}^d$ so that they touch in such a way that all their pairwise intersections are $(d-1)$-dimensional? This is an old and very natural question. We shall call $f(d)$ the answer to this problem. In 1956, Bagemihl [1] posed the following conjecture.

Conjecture (Bagemihl, 1956)**.** *The maximal number of pairwise touching d-simplices in a configuration in $\mathbb{R}^d$ is*

$$f(d) = 2^d.$$

The conjecture is verified for dimensions $d \leq 3$. Zaks [7] showed that $f(d) \geq 2^d$. For the upper bound, Perles [6] showed that $f(d) < 2^{d+1}$. Very recently, Kisielewicz [3] announced that $f(d) \leq 2^{d+1} - \omega(1)$. He obtained this bound by relating $f(d)$ to an extremal problem for finite sets which we describe below.

Two words $u, v \in \{0, 1, *\}^n$ are called *neighborly* if there is precisely one i such that $\{u_i, v_i\} = \{0, 1\}$. Two words $u, v \in \{0, 1, *\}^n$ are a *twin pair* if for some $i \in [n]$ we have $\{u_i, v_i\} = \{0, 1\}$ and $u_j = v_j$ for all $j \neq i$. We say a family $\mathcal{G} \subseteq \{0, 1, *\}^n$ is

- a *d-code* if $|\{i \in [n] : v_i \neq *\}| = d$ for all $v \in \mathcal{G}$;
- *neighborly* if every two words in $\mathcal{G}$ are neighborly;

Let $g(d)$ denote the maximal size of a neighborly d-code without twin pairs. Kisielewicz showed that $f(d) \le g(d+1)$ and $g(d) \le 2^d - \omega(1)$ (hence $f(d) \le 2^{d+1} - \omega(1)$).

Kisielewicz and Przeslawski [4, 5] inductively construct a neighborly d-code without twin pairs as follows. We begin with $\mathcal{G}_2 = \{00*, *10, 1*1\}$. Let $m = 2^d - 1$ and define

$$\mathcal{G}_{d+1} = \{w *^m 0 : w \in \mathcal{G}_d\} \cup \{*^m w1 : w \in \mathcal{G}_d\},$$

where $*^m$ is the word consisting of m stars. It is not hard to show that $\mathcal{G}_d \subseteq \{0,1,*\}^{2^d-1}$ is a neighborly d-code without twin pairs of size $|\mathcal{G}_d| = 3 \cdot 2^{d-2}$. This implies $g(d) \ge 3 \cdot 2^{d-2}$.

Conjecture (Kisielewicz, 2023)**.** *For $d \ge 2$, $g(d) = 3 \cdot 2^{d-2}$. Moreover, up to isomorphism, $\mathcal{G}_d$ is the unique extremal example.*

The conjecture, if true, would imply $f(d) \le g(d+1) = \frac{3}{2} \cdot 2^d$.

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Problems

Proposed by Ethan Patrick White, University of Illinois at Urbana-Champaign

Turán numbers of Wickets

The Turán number of a linear 3-uniform hypergraph F is the maximum number of edges of a 3-uniform linear hypergraph not containing a subgraph isomorphic to F, we denote this by $ex_L(n, F)$. For all F with at most 5 edges, Gyárfás and Sárközy showed that either $ex_L(n, F) = o(n^2)$ or $ex_L(n, F) \geq n^2/9$, with one exception: the wicket! The wicket W has nine vertices arranged in a grid, and edges on each of the rows and columns, minus the middle column. Earlier this year, Solymosi answered their question and proved that $ex_L(n, W) = o(n^2)$. The deletion method shows $ex_L(n, W) = \Omega(n^{3/2})$. He poses the problem of determining $ex_L(n, W)$ more precisely.

Question. *Determine* $ex_L(n, W)$.

Avoiding triangles in grids

The largest subset of the grid $[N] \times [N]$ that does not contain an axis-parallel isosceles triangle, often called a *corner* has size $o(N^2)$. An elegant proof of this is due to Solymosi using the triangle removal lemma, and reasonably tight upper and lower bounds exist. Two variations on this problem are the following.

Problem (Shkredov, Solymosi [2])**.** *Determine the smallest subset* $S \subset \mathbb{F}_p \times \mathbb{F}_p$ *such that adding any point to* S *creates an isosceles right triangle. This value is denoted* $sat(\mathbb{F}_p \times \mathbb{F}_p, C)$.

Shkredov and Solymosi show $p/\sqrt{3} \leq sat(\mathbb{F}_p \times \mathbb{F}_p, C) \leq p$, and ask in particular for a saturated subset of size at most $p - 1$.

Problem. *What is the largest subset of* $[N] \times [N]$ *that does not contain an isosceles right triangle? What is the largest subset of* $[N] \times [N]$ *that does not contain an isosceles triangle (tilted corners)?*

For both of these questions, it would be interesting to find a construction with $\Omega(N^{1+c})$ points, or upper bounds of the form $O(N^{2-c})$ for any $c > 0$.

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Problem

Proposed by Michael Wigal, University of Illinois at Urbana-Champaign

It is well-known that in a connected graph, every pair of longest paths must pairwise intersect. It remains an open question on whether every three longest paths always have nonempty intersection in a connected graph.

Fuglede's conjecture in finite fields

Proposed by Tao Zhang, Zhejiang Lab

Let p be a prime number.

Definition. *A subset $A \subset \mathbb{F}_p^d$ is called spectral if there exists a subset $B \subset \mathbb{F}_p^d$ such that*

$$\{\chi_a(x) = e^{\frac{2\pi i x \cdot a}{p}} : a \in B\}$$

forms an orthogonal basis of the complex vector space $L^2(A)$. We then say B is a spectrum of A and (A, B) is a spectral pair.

Definition. *A subset $A \subset \mathbb{F}_p^d$ is said to tile $\mathbb{F}_p^d$ by translations if there exists a subset $T \subset \mathbb{F}_p^d$ such that all elements in $\mathbb{F}_p^d$ can be uniquely represented as*

$$a + t,$$

where $a \in A$ and $t \in T$.

The interest in the connection between spectral sets and tiles arises from a conjecture of Fuglede, which states that for a subset $\Omega \subset \mathbb{R}^d$ with positive and finite Lebesgue measure, $L^2(\Omega)$ has an orthogonal basis of exponentials if and only if Ω tiles $\mathbb{R}^d$ by translations. In 2004, Tao disproved the Fuglede's conjecture in $\mathbb{R}^d$ for dimensions $d \geq 5$ by lifting a non-tiling spectral set in $\mathbb{F}_3^5$ to Euclidean space. This sparked interest in the discrete setting of Fuglede's conjecture. We summarize known results of Fuglede's conjecture in finite fields.

Theorem. *Fuglede's conjecture is false in following groups:*

1. $\mathbb{F}_p^d$ *for* $d \geq 5$*;*
2. $\mathbb{F}_p^4$ *for odd prime* p*.*

And Fuglede's conjecture holds in following groups:

1. $\mathbb{F}_p^2$*;*
2. $\mathbb{F}_2^4$*;*
3. $\mathbb{F}_p^3$ *for* $p = 2, 3, 5, 7$*.*

Hence, the only remaining case is the following question.

Question. *Fuglede's conjecture holds or not in $\mathbb{F}_p^3$ for $p \geq 11$?*