Since flag algebras can be defined for various discrete structures, we can transform an extremal problem into a problem expressible in terms of densities of some structures and use flag algebras on them.

For instance, consider the following problem. Erdős, Faudree and Rousseau conjectured in 1992 that any graph on n vertices and $\lceil \frac{n^2}{4} \rceil + 1$ edges contains asymptotically at least $\frac{2}{9}n^2$ edges that occur in a C_5. It occurred that the constant is wrong as there is a construction having asymptotically only $\frac{2+\sqrt{2}}{16}n^2 \approx 0.2134n^2$ edges in C_5. Despite the fact that an occurrence in a C_5 cannot be expressed in terms of graph densities, as well as having exactly $\lceil \frac{n^2}{4} \rceil + 1$ edges, we can solve this problem using flag algebras. In order to express this problem in a density setting, we consider graphs with edges colored red and blue, and additionally assume that there is no C_5 containing a blue edge. For such graphs we prove that if  $+$  $\geq \frac{1}{2}$, then

 $\left(8\right.$$\left. - 2 - \sqrt{2}\right) \geq 0.$

This inequality means that if we consider any sequence of graphs with $\lceil \frac{n^2}{4} \rceil + 1$ edges, then either the density of edges occurring in a C_5 is at least $\frac{2+\sqrt{2}}{8}$ as needed, or there are $o(n^3)$ triangles. The remaining case can be solved using stability of triangle-free graphs.

As a second example, we show how to determine Ramsey numbers using flag algebras. This is based on a paper by Lidický and Pfender. For graphs $G_1, \ldots, G_k$ the Ramsey number $R(G_1, \ldots, G_k)$ is the smallest n for which in every edge coloring with k colors of a complete graph K_n there exists copy of G_i with all edges colored i or some $i \in [k]$. In order to express it as a density problem, we consider edge colorings of graphs using k colors forbidding for each $i \in [k]$ appearance of a complete graph having G_i in color i, as well as a triple of vertices x, yz with x, y non-adjacent and z connected to x and y in different colors. This setting means that non-adjacent vertices forms a groups and between any two groups we have all edges in the same color, i.e., we work on blow-ups of complete graphs with colored edges. Note that if there exists a coloring of a complete graph K_n without a copy of G_i in color i, then by considering its bow-ups we obtain a sequence of graphs in the considered setting with the density of non-edges equal to $\frac{1}{n}$. In particular, if we show that the density of non-edges in our setting is at most some δ, then it implies that

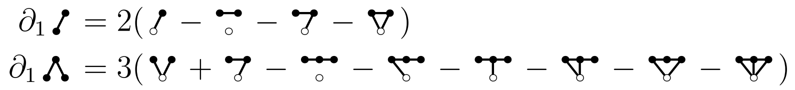
$$R(G_1, \ldots, G_k) \leq \frac{1}{\delta} + 1.$$

As an example, we sketch the proof of $R(K_3, K_3) = 6$. Obviously $R(K_3, K_3) \geq 6$ as there is an edge coloring of K_5 using two colors without monochromatic triangles, so we need to show that $R(K_3, K_3) < 7$. Consider the setting described in the previous paragraph, so graphs with edges colored in two colors without , , ,  and . We take $n = 4$ and consider two rooted graphs, each consisting of an edge in one of the colors, and flags with one unlabeled vertex. Using an SDP solver returns semidefinite matrices, that lead to the bound  $\geq 0.17 > \frac{1}{6}$. This implies that $R(K_3, K_3) < 7$ as wanted.

2.6 Differential method

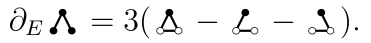
Suppose we have a graph theory problem to maximize some function $f(H_1, H_2, \ldots, H_k)$ of densities of graphs $H_1, \ldots, H_k$. If a graphon W realizes the maximum, then a small perturbation of W should not give a higher value of f. This is the main idea of the differential method developed by Razborov. Here we provide only an intuitive explanation.

Informally speaking, as a small perturbation we may consider for instance cloning or removing a particular vertex. The change of the density of a graph H cased by such a small perturbation using a fixed vertex is the difference between the density of H containing the fixed vertex and the global density of H. We denote this difference multiplied by $|H|$ because of normalizing issues by $\partial_1 H$. In particular we have



For instance, if our problem is just to maximize the density of H, the extremal graphon needs to satisfy $\partial_1 H = 0$ for every possible placement of the root. It is tempting to think that one can just add $[\![\partial_1 H]\!] = 0$ to the set of constraints satisfied by an extremal graphon W, which can then be used to derive an upper bound on $d(H, W)$. Unfortunately, $[\![\partial_1 H]\!] = 0$ holds as an identity in the algebra, so such addition is not helpful. In order to benefit from this approach, one needs to multiply $\partial_1 H$ by some rooted expression g and add an assumption $[\![g \cdot \partial_1 H]\!] = 0$. For a suitable choice of g it can provide a profitable constraint.

A different small perturbation that one may consider is to remove an edge between two fixed vertices. This leads to defining $\partial_E H$ as the possible gain in the density of H under removing a fixed edge. Because of normalizing issues, this needs to be multiplied by $\binom{|H|}{2}$. In particular,



Therefore, if our problem is to maximize the density of H, the extremal graphon needs to satisfy $\partial_E H \leq 0$ as it should not be possible to gain by removing the fixed edge. Similarly as before, we can consider any non-negative expression g on a rooted edge and add to our constraints on the extremal graphon an assumption $[\![g \cdot \partial_E H]\!] \leq 0$.

In general, assume a function $f(H_1, H_2, \ldots, H_k)$ of densities of graphs $H_1, \ldots, H_k$ attains its maximum in some point $a = (a_1, \ldots, a_k)$ and f is continuously differentiable in some open neighborhood of the point a. Then for the extremal graphon it must hold

$$\frac{\partial f}{\partial H_1}(a) \cdot \partial_1 H_1 + \frac{\partial f}{\partial H_2}(a) \cdot \partial_1 H_2 + \ldots + \frac{\partial f}{\partial H_k}(a) \cdot \partial_1 H_k = 0 \tag{9}$$

for every placement of the root. And also

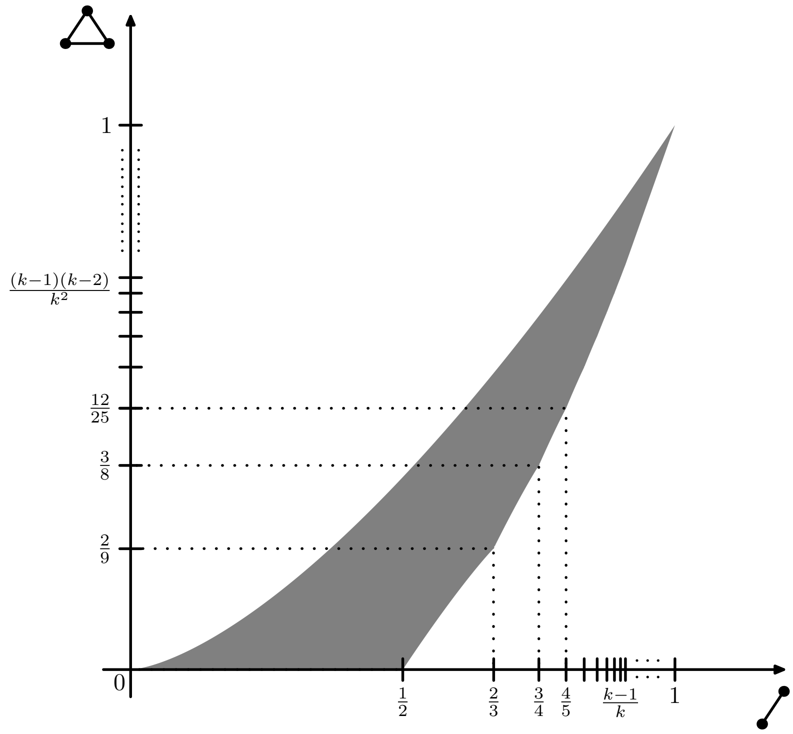
$$\frac{\partial f}{\partial H_1}(a) \cdot \partial_E H_1 + \frac{\partial f}{\partial H_2}(a) \cdot \partial_E H_2 + \ldots + \frac{\partial f}{\partial H_k}(a) \cdot \partial_E H_k \leq 0 \tag{10}$$

for every placement of the rooted edge.

To observe how one can use this approach, we consider the following problem. For a given density of edges, determine the minimum possible density of triangles. The question appears in print in a paper of Erdős from 1955, but since the first non-trivial case was solved by Rademacher in 1941, it is now known as the Erdős-Rademacher problem.

Using a simple Cauchy-Schwarz inequality (Theorem 5) one can prove that always $\triangle \geq \diagup(2\diagup - 1)$. This is equality for $\diagup = 1 - 1/t$ and any integer $t \geq 1$ as this bound is achieved in a balanced complete multipartite graph. There were many improvements to the above bound. In particular, Bollobás in 1976 proved the piecewise linear bound between the points with $\diagup = 1 - 1/t$ for each integer $t \geq 2$. For $\diagup \in \left[1 - \frac{1}{t}, 1 - \frac{1}{t+1}\right]$ the optimal construction is to consider a blow-up of K_{t+1} with t blobs of equal size and one smaller blob. This leads to the following optimal bound

$$\triangle \geq \frac{(t-1)\left(t - 2\sqrt{t(t-(t+1)\diagup)}\right)\left(t + \sqrt{t(t-(t+1)\diagup)}\right)^2}{t^2(t+1)^2}. \tag{11}$$



The problem was finally solved by Razborov in 2008 using flag algebras. In 2011 Nikiforov determined similar function for the density of K_4, while in 2016 Reiher determined it for all bigger complete graphs. In 2017 Pikhurko and Razborov provided the description of the extremal graphs for the triangle problem. While in 2020 Liu, Pikhurko and Staden gave an exact solution for the number of triangles depending on the number of edges for all large graphs whose edge density is bounded away from 1, together with a description of the extremal graphs.

To present how one can use the differential method, we sketch a proof of the bound (11) for $t = 2$, which means

$$\triangle \geq \frac{(1 - \sqrt{4 - 6\diagup})(2 + \sqrt{4 - 6\diagup})^2}{18} \text{ for } \diagup \in \left[\frac{1}{2}, \frac{2}{3}\right].$$

Denote by $g(\diagup)$ the right-hand side of the above inequality. Let $f(\diagup, \triangle) = g(\diagup) - \triangle$ and a be the point in $\left[\frac{1}{2}, \frac{2}{3}\right] \times [0, 1]$, where the function f attains its maximum. We need to show that $f(a) \leq 0$. If the maximum is attained for $\text{E} = \frac{1}{2}$ or $\frac{2}{3}$, then the bound follows from the mentioned earlier result. Thus, f is continuously differentiable in the open neighborhood of a. Using (9) multiplied by E_1 and averaged we obtain

$$[\![g'(\text{E})\, \text{E}_1 \cdot \partial_1 \text{E} - \text{E}_1 \cdot \partial_1 \text{T}]\!] = 0,$$

which implies

$$2g'(\text{E})\, [\![\text{E}_1^{\,2}]\!] - 2g'(\text{E})\, \text{E}^2 - 3\, [\![\text{E}_1 \cdot \text{T}_1]\!] + 3\, \text{E} \cdot \text{T} = 0. \tag{12}$$

While using (10) multiplied by P_E and averaged we get

$$[\![g'(\text{E})\, \text{P}_E \cdot \partial_E \text{E} - \text{P}_E \cdot \partial_E \text{T}]\!] \leq 0,$$

which leads to

$$-\frac{1}{3} g'(\text{E})\, \text{P} + 3\, [\![\text{P}_E \cdot \text{T}_E]\!] \leq 0.$$

Subtracting this inequality from (12) we obtain

$$g'(\text{E}) \left(2\, [\![\text{E}_1^{\,2}]\!] + \frac{1}{3} \text{P} \right) - \left(3\, [\![\text{E}_1 \cdot \text{T}_1]\!] + 3\, [\![\text{P}_E \cdot \text{T}_E]\!] \right) + 3\, \text{E} \cdot \text{T} \geq 2g'(\text{E})\, \text{E}^2. \tag{13}$$

By simple computations we have

$$2\, [\![\text{E}_1^{\,2}]\!] + \frac{1}{3} \text{P} = \text{T} + \text{E},$$

while comparing coefficients by all 4 graphs on 4 vertices containing a triangle we obtain the inequality

$$3\, [\![\text{E}_1 \cdot \text{T}_1]\!] + 3\, [\![\text{P}_E \cdot \text{T}_E]\!] \geq 2\, \text{T}.$$

Thus, applying the above two expressions to (13) we obtain

$$\text{T}\, (g'(\text{E}) + 3\text{E} - 2) \geq (2\text{E}^2 - \text{E}) g'(\text{E}). \tag{14}$$

Since the function $g(x)$ for $x \in (\frac{1}{2}, \frac{2}{3})$ satisfies the differential equation

$$g(x)(g'(x) + 3x - 2) = (2x^2 - x) g'(x)$$

and $g'(x) \geq 1$, so $g'(\text{E}) + 3\text{E} - 2 > 0$, the inequality (14) implies $\text{T} \geq g(\text{E})$ as wanted.