A mini-course on the hypergraph container method

Exercises

Problem 1. Show that, for every $G \subseteq K_n$ and all H,

$$\mathrm{ex}(G, H) \geqslant \frac{\mathrm{ex}(n, H)}{\binom{n}{2}} \cdot e(G).$$

Problem 2. Prove Hoeffding's theorem: For all real $0 < p \leqslant q \leqslant 1$ and positive integer N,

$$\Pr\big(\mathrm{Bin}(N, p) \geqslant qN\big) \leqslant \exp\big(-N \cdot I_p(q)\big), \tag{1}$$

where $I_p(q) \coloneqq q \log \frac{q}{p} + (1-q) \log \frac{1-q}{1-p}$. Further, use (1) to conclude that:

(i) $\Pr\big(\mathrm{Bin}(N, p) \leqslant (1-\delta)Np\big) \leqslant \exp\Big(-\frac{\delta^2 Np}{2}\Big)$ for every $\delta \in [0, 1]$ and

(ii) $\Pr\big(\mathrm{Bin}(N, p) \geqslant (1+\delta)Np\big) \leqslant \exp\Big(-\frac{\delta^2 Np}{2(1+\delta/3)}\Big)$ for every $\delta > 0$.

Hint: Suppose that $X \sim \mathrm{Bin}(N, p)$. Since the function $x \mapsto e^{\lambda x}$ is strictly increasing for every $\lambda > 0$, we have $\Pr(X \geqslant qN) = \Pr(e^{\lambda X} \geqslant e^{\lambda q N})$. Use Markov's inequality to bound the above probability.

Problem 3. Assume that n is divisible by four and consider the following model of random triangle-free graphs. Fix a partition $[\![n]\!] = A \cup B$ and place an arbitrary matching M of $n/4$ edges in A. Make every vertex in B adjacent to exactly one randomly chosen endpoint of each edge of M, so that the resulting graph has precisely $n^2/8 + n/4$ edges. Show that almost all such graphs (out of the total of $2^{n^2/8}$) are maximal triangle-free.

Problem 4. Prove that the sequence $n \mapsto \mathrm{ex}(n, H)/\binom{n}{2}$ is nonincreasing for every graph H. (In particular, $\pi_H \coloneqq \lim_{n\to\infty} \mathrm{ex}(n, H)/\binom{n}{2}$ exists for every graph H and $\pi_H \in [0, 1]$.)

Problem 5. Prove that, for every integer $r \geqslant 2$, there exists a K_4-free graph G satisfying $G \longrightarrow (K_3)_r$. To this end, consider the random graph $G_{n,p}$ for an appropriately chosen density p. What is the probability that $G_{n,p} \not\supseteq K_4$? (Hint: Use Harris's inequality.) What is the probability that $G \longrightarrow (K_3)_r$?

Problem 6. Prove that, for every integer $r \geqslant 2$, there exist constants $\delta > 0$ and K such that the following holds: If $p \geqslant K n^{-1/2} \log n$, then a.a.s. $G \longrightarrow (K_3)_r$ for every $G \subseteq G_{n,p}$ with $e(G) \geqslant (1-\delta)\binom{n}{2}p$. Use this fact to give another proof of the existence of a K_4-free graph that is Ramsey for K_3.

Problem 7. Prove that, for every $r \geqslant 2$, there exists a constant K such that $G_{n,p} \longrightarrow (K_3)_r$ a.a.s. whenever $p \geqslant K n^{-1/2}$.

Problem 8. Derive the following stability theorem for almost-triangle-free graphs from Simonovits's stability theorem and the triangle-removal-lemma: For every positive δ, there exists a positive ε such that every n-vertex graph with fewer than εn^3 triangles and at least $\mathrm{ex}(n, K_3) - \delta n^2/6$ edges can be made bipartite by removing from it at most δn^2 edges.

Problem 9. Prove that, for every $\delta > 0$, there exists a consntant K such that, if $m \geqslant Kn^{3/2}$, then almost every K_3-free graph with vertex set $[\![n]\!]$ and m edges can be made biparte by removing from it at most δm edges.

Problem 10. Show that, for every graph H with at least two edges, there is a family of containers for the set of H-free graphs with vertex set $[\![n]\!]$ with fingerprints of size $b = O(n^{2-1/m_2(H)})$.

Problem 11. Show that, for every graph H with at least two edges, every $\delta > 0$ and all $r \geqslant 2$, there exists a constant K such that the following holds if $p \geqslant Kn^{-1/m_2(H)}$ (respectively, if $m \geqslant Kn^{2-1/m_2(H)}$):

(i) $\mathrm{ex}(G_{n,p}, H) = \mathrm{ex}(n, H) \cdot p \pm \delta n^2 p$ a.a.s.;

(ii) $|\mathcal{F}_{m,m}(H)| \leqslant \binom{\mathrm{ex}(n,H)+\delta n^2}{m}$;

(iii) $G_{n,p} \longrightarrow (H)_r$ a.a.s.;

(iv) If $\chi(H) > 2$, then a.a.s. every largest H-free subgraph of $G_{n,p}$ can be made $(\chi(H) - 1)$-partite by removing from it at most $\delta n^2 p$ edges. (The case $\chi(H) = 2$ follows easily from (i) and is much less interesting.)

(v) If $\chi(H) > 2$, then almost every $F \in \mathcal{F}_{n,m}(H)$ can be made $(\chi(H)-1)$-partite by removing from it at most δm edges. (Note that, in the case $\chi(H) = 2$, this statement is clearly false.)

Problem 12. Show that for (i), (ii), and (iv) above, the assumption that $p \geqslant Kn^{-1/m_2(H)}$ (respectively, $m \geqslant Kn^{2-1/m_2(H)}$) is optimal, up to the constant K.

Problem 13. Show that the hypergraph defined in the proof of the existence of graphs graphs G satisfying $G \longrightarrow_{\mathrm{ind}} (H)_r$ satisfies the assumption of the container theorem with $b = O(n^{2-1/m_2(K_{v_H})})$.

Problem 14. For a set $P \subseteq \mathbb{R}^2$, let $f(P)$ denote the largest size of a subset of P containing no collinear 3-tuples. Prove that, if P contains no collinear 4-tuples, then $f(P) \geqslant (\sqrt{2}-o(1))\cdot|P|^{1/2}$, as $|P| \to \infty$.

Problem 15. Let $\pi\colon \mathbb{R}^3 \to \mathbb{R}^2$ be the orthogonal proejection on a random hyperplane thorugh the origin. Prove that, for every triple $x, y, z \in \mathbb{R}^3$ that are not collinear,

$$\Pr\big(\pi(x),\ \pi(y),\text{ and } \pi(z) \text{ are collinear}\big) = 0.$$

Problem 16. Prove that every set $A \subseteq [\![m]\!]^3$ with at least $100m^2$ points contains at least $c|A|^4/m^6$ collinear triples, for some absolute positive constant c.