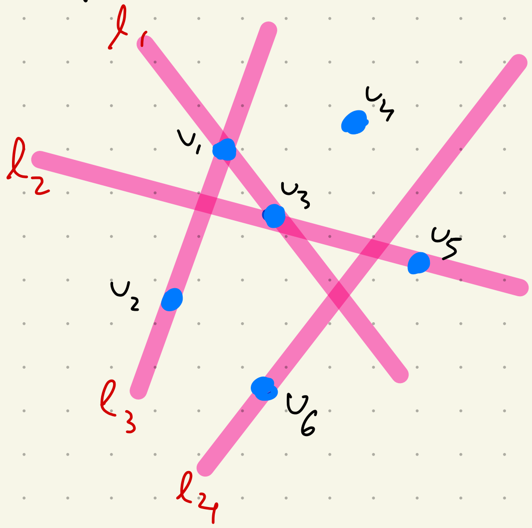
Lecture 32

§ C_6-free incidence graphs.

Given a pts/lines (P / L) arrangement on the plane $\mathbb{R}^2$,

an incidence is a pair $(p, \ell) \in P \times L$

s.t. $p \in \ell$.



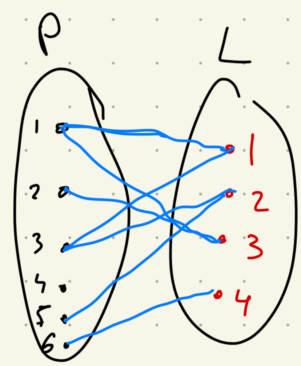
Ex $|P| = 6$, $|L| = 4$

$\#\text{incidence} = I(P, L) = 7$

The pt/line incidence graph for an arrangement (P, L)

is a bip. graph w/ parts P & L

and $p\ell \in G \iff p \in \ell$



- $|I(P, L)| = e(G)$
- $\deg(pt) = \#$ lines thru it.

Rmk pt/line incidence gr. is C_4-free

Thm (Szemerédi-Trotter)

$\forall$ pt/line arrangement w/ $|P| = |L| = n$

$\Rightarrow |I(P, L)| = O(n^{4/3})$

Rmk $ex(n, C_4) = \Theta(n^{3/2})$. Here much better bound for "geometric" graphs

Recall $ex(n, C_6) = \Theta(n^{4/3})$.

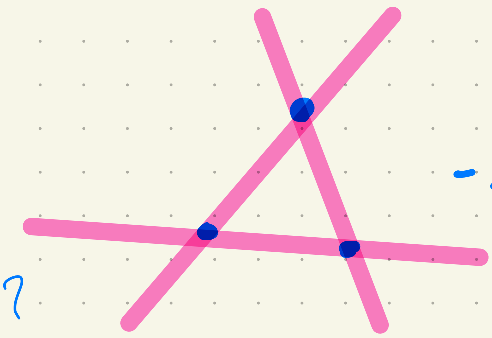
Thm (Solymosi: Dense arr. locally dense)

Given an (n,n)-vx pt/line incidence gr. G,

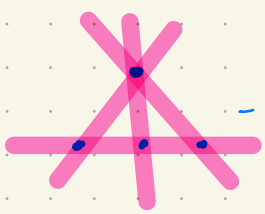
if G is C_6-free $\Longrightarrow$ $e(G) = o(n^{4/3})$

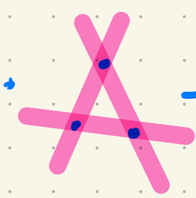
Rephrase Given pt/line arrangement (P, L) with no 3 pts in general position pairwise lying on some lines.

$\Longrightarrow$ $|I(P,L)| = o(n^{4/3})$

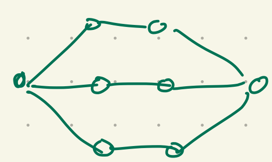


-free

Open. 3-fan  -free $\Rightarrow o(n^{4/3})$?

 -free $\Rightarrow O(n^{4/3 - c})$?

incidence gr is $\Theta_{3,3}$-free



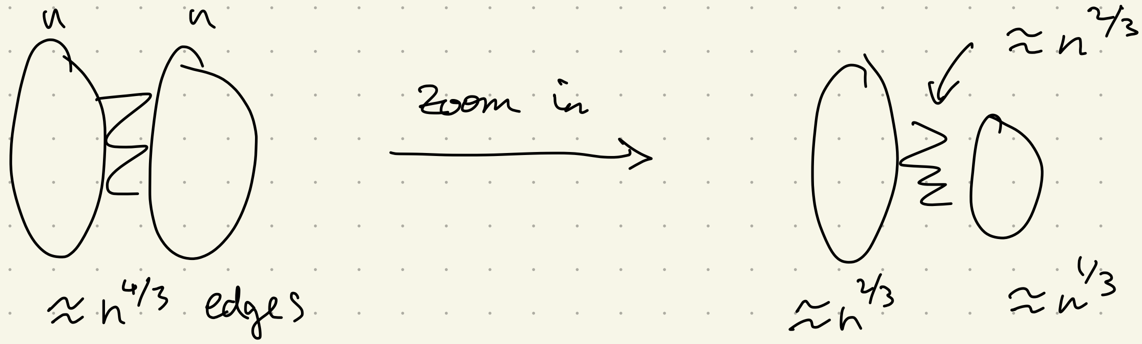
Two ingredients: - partition lem.
- removal lem.

Idea: • graphs obtained from geometric settings tend to have their edges clustered

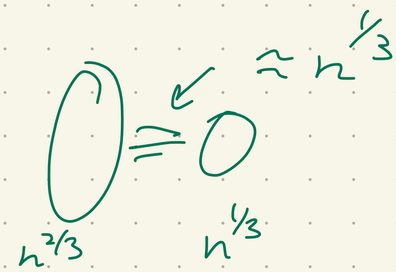
$\Longrightarrow$ Zoom into exceptionally dense asymm. bip. part.

(Matoušek's partition)

Then use removal lem.



- If we zoom in randomly, $\Rightarrow$



Matoušek partition ($d=2$)

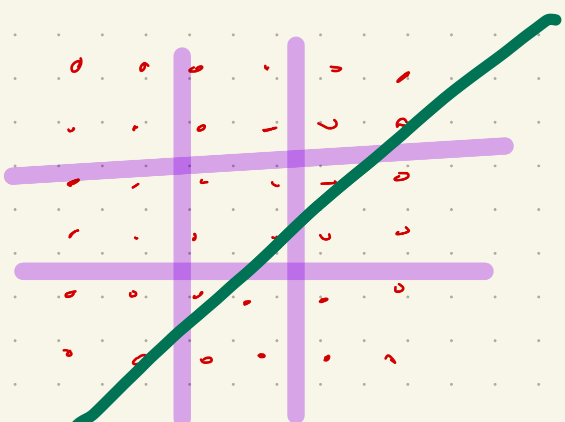
Given
- $P \subseteq \mathbb{R}^2$ n pts
- $2 \le r < n$

$\Longrightarrow$ $\exists$ partition

$P = D_1 \cup \cdots \cup D_t$

- $n/r \le |D_i| \le 2n/r$
- any line passes through $O(\sqrt{r})$ sets

Intuitive: imagine pts $\approx$ grid pts



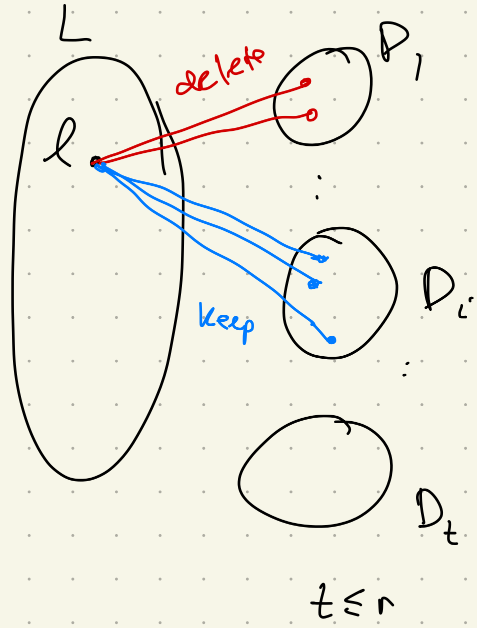
$\sqrt{r}$ by $\sqrt{r}$ cutting

Pf Suppose the incidence gr G has $\gtrsim c n^{4/3}$ edges.

Take $\beta \ll c$ and set $r = \beta n^{2/3}$.

- Apply M. partition $\Rightarrow$ $P = D_1 \cup \cdots \cup D_t$

every vx $\ell \in L$ has neighbors in $O(\sqrt{r})$ different D_i's



1) $\forall\ \ell \in L\ \ \forall\ D_j$

if $\deg(\ell, D_j) \le 2$

deletes edges from ℓ to D_j.

$$\#\text{edges deleted} \le n \cdot 2 \cdot O(\sqrt{r})$$

$$= O(\beta^{1/2} \cdot n^{4/3}) \underset{\substack{\uparrow \\ \beta << c}}{\le} \frac{1}{2} e(G)$$

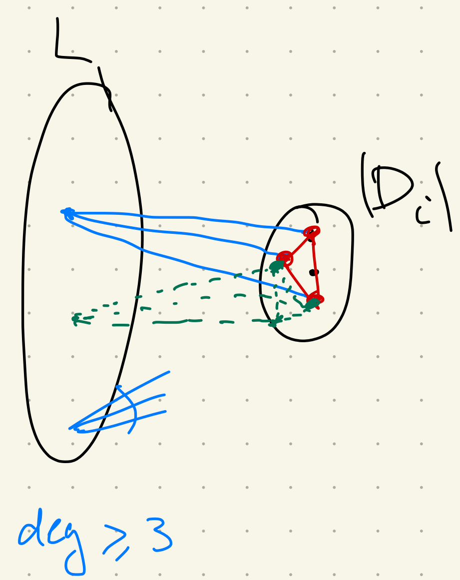
By averaging, $\exists\ i \in [t]$, s.t.

$$\#\text{ edges from } L \text{ to } D_i \ge \frac{\frac{1}{2}e(G)}{t} \ge \frac{\frac{1}{2}e(G)}{r}$$

$$= \frac{c}{2\beta} n^{2/3}$$

- Look at $[L, D_i]$

$n/r \le |D_i| \le 2n/r \approx \frac{1}{\beta} n^{1/3}$



Build an auxiliary gr. Γ w./

$V(\Gamma) = D_i$

$E(\Gamma)$: for each $\ell \in L$ w./

$d_\ell = \deg(\ell, D_i) \ge 3$

put $\lfloor \frac{d_\ell}{3} \rfloor$ vx-disj Δs in

$N(\ell, D_i)$

- In Γ, call a triangle L-triangle if it comes from some $\ell \in L$

all L-triangles are edge-disjoint as G is C_4-free

$$\Rightarrow \underset{\text{in } \Gamma}{\#\text{ edge-disj } \Delta\text{s}} \geq \#\,L\text{-triangles} = \sum_{\ell} \lfloor d_\ell/3 \rfloor$$

since $d_\ell \geq 3$ $\quad \geq \sum_\ell d_\ell/6$

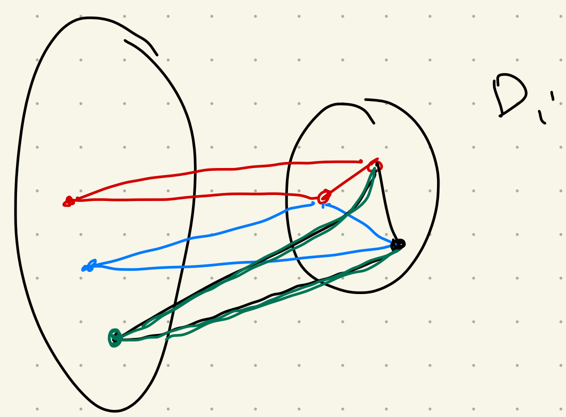
$$\geq \frac{1}{6}\frac{c}{2\beta} n^{2/3} = \Omega\left(|D_i|^2\right)$$

- Removal lem $\Rightarrow$ Γ has $\Omega(|D_i|^3)$ Δs

$$\Omega(|D_i|^3) = \Omega(n) \gg O(n^{2/3}) \geq \#\,L\text{-triangles}$$

$\Rightarrow$ $\exists$ a non-L-triangle in Γ.

$\Rightarrow$ C_6 in G $\lightning$.



Def: $r(G,G) = \min N$ s.t. any 2-edge-col. of K_N

$$\Longrightarrow \text{mono}X.\ G.$$

Classical: $2^{n/2} < r(K_n, K_n) < 4^n$

Thm (Chvátal - Rödl - Szemerédi - Trotter)

- $d \in \mathbb{N}$, $\Delta(G) \le d$

$$\Longrightarrow r(G,G) = O_d(|G|)$$

- Multicolor ver. of reg. lem.

For a k-edge-col. gr. G, a ptt. $V(G) = V_1 \cup \cdots \cup V_r$ is an ε-reg. ptt. if

- $\forall\, ij \in \binom{[r]}{2}$, $\big||V_i| - |V_j|\big| \le 1$;
- for all but $\le \varepsilon\binom{r}{2}$ pairs $ij \in \binom{[r]}{2}$, (V_i, V_j) is ε-reg. in every color.

Multicol. Reg. Lem. $\forall\, \varepsilon > 0$, $k, m \in \mathbb{N}$, $\exists\, M = M(\varepsilon, m, k)$ s.t. every k-edge-col. gr. G w/ $n \ge m$ vxs admits an ε-reg. ptt $V(G) = V_1 \cup \cdots \cup V_r$, where $m \le r \le M$

Reduced gr. $\longleftrightarrow$ reg. p.tt.

Only difference: assign majority col. to $E(R)$

Thm (Brook's thm) $\forall G \Rightarrow \chi(G) \leq \Delta(G)+1$.

Pf (Ch-R-Sz-Tr)

Let $m \geq 5\, r(K_{d+1}, K_{d+1})$, $\varepsilon = 1/m$ and

$$C := \frac{2M}{\left(\frac{1}{2}-\varepsilon\right)^d}.$$

Let $N \doteq C|G|$ and fix an arbitrary 2-edge-col. of K_N.

Goal: Find a monoχ. copy of G.

Apply multicol. reg. lem. to the given 2-edge-col. K_N and let R be the corresp. reduced gr. col. a 2-edge-col. indicating the majority color.

- As $\leq \varepsilon\binom{r}{2}$ irreg pairs, R is almost complete:

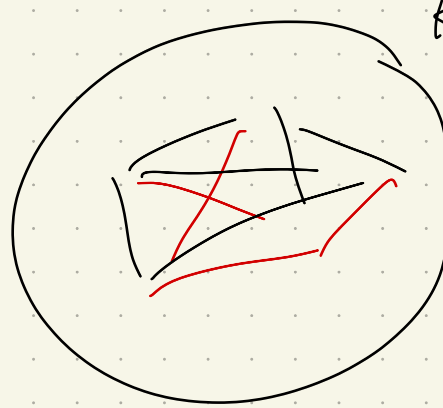
$$e(R) \geq (1-\varepsilon)\binom{r}{2} > \left(1-\frac{1}{m/3-1}\right)\frac{r^2}{2}$$

- Turán's thm $\Rightarrow$ R contains a clique $K_{m/3}$

- $m/3 > r(K_{d+1}, K_{d+1})$

  $\Rightarrow$ $\exists$ mono χ. $K_{d+1} \subseteq K_{m/3} \subseteq R$

$K_{m/3} \subseteq R$



- K_{d+1} is a hom. image of G

  as $\Delta(G) \leq d$ by Brook's thm

- Embedding lem: mono χ $K_{d+1} \subseteq R \rightsquigarrow$ mono χ G in K_N.