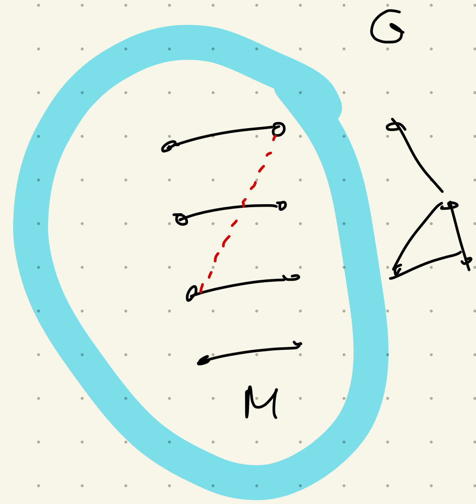
Lecture 31



§ Induced matching

<u>Def</u>: M is an induced matching in G

if $E(G[V(M)]) = E(M)$

<u>Thm</u> n-vx G union of n induced matchings, $E(G) = \bigcup_{i=1}^{n} M_i$

$\Longrightarrow$ $e(G) = o(n^2)$

$\uparrow$ induced matching

• Application $\begin{cases} 1) \Rightarrow (6,3)\text{-thm} \leftarrow \\ 2) \Rightarrow \text{Roth's thm} \quad \text{Exer} \end{cases}$

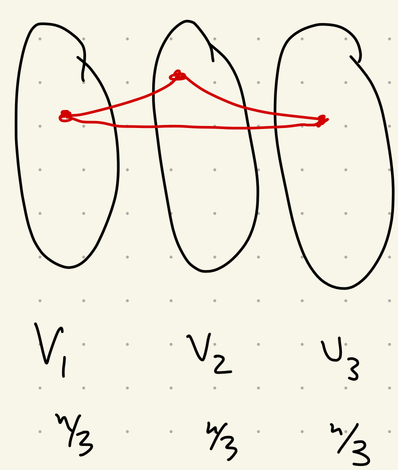
<u>Pf</u> (Induced matching thm $\Rightarrow$ $(6,3)$-thm)

<u>NTS</u>: $\forall \mathcal{H}$ 3-unif $(6,3)$-free $\Rightarrow$ $e(\mathcal{H}) = o(n^2)$

• By passing to a subgr. w/ highest ave. deg., we may assume $\mathcal{H}$ is linear.

• Take a random equipartition

$V(\mathcal{H}) = V_1 \cup V_2 \cup V_3$



$\Longrightarrow$ expected # edges in $\mathcal{H}[V_1, V_2, V_3]$

$= \frac{2}{9} e(\mathcal{H})$

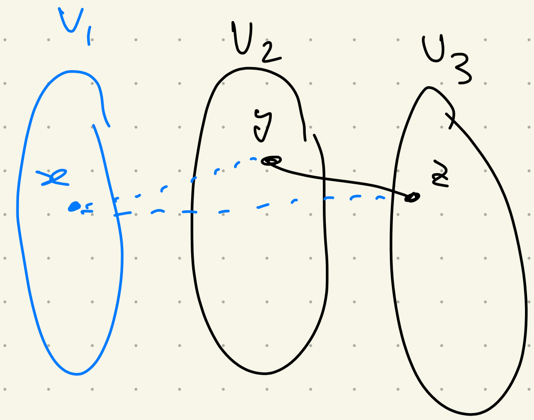
$\Longrightarrow$ $\exists$ an equiptt. V_1, V_2, V_3 w/

cross edges $e_{cr} \geq \frac{2}{9} e(\mathcal{H})$

Goal: $e_{cr} = o(n^2)$

- Define an auxiliary (shadow) bipartite graph G on $V_2 \cup V_3$.

$E(G) = \{yz : xyz \in E(\mathcal{H}),\ x \in V_1, y \in V_2, z \in V_3\}$



G = union of all link gr. of vxs in V_1

- Write M_v = link of $v \in V_1$

$\mathcal{H}$ linear $\Rightarrow \begin{cases} \cdot\ M_v \text{ matching } \forall v \in V_1 \\ \cdot\ e(G) = \sum_{v \in V_1} |M_v| = e_{cr} \end{cases}$

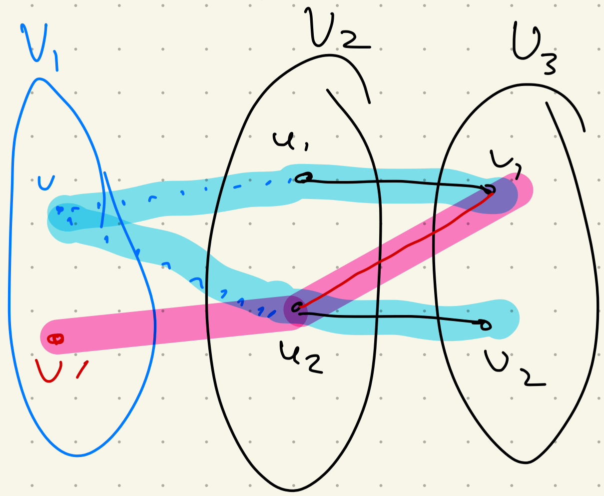
matchings are edge disj

Suffices to show $e(G) = o(n^2)$

To apply IMT, left to show that M_v is an induced matching $\forall v \in V_1$.

- Suppose not: $\exists u_1v_1, u_2v_2 \in M_v$ s.t. there is another edge say u_2v_1 wlog

$\Rightarrow u_2v_1 \in M_{v'}$ for some $v' \neq v$, $v' \in V_1$



$\Rightarrow$ $(G,3)$-subgr. in $\mathcal{H}$ $\lightning$. ☺.

<u>Pf</u> (Induced matching thm)

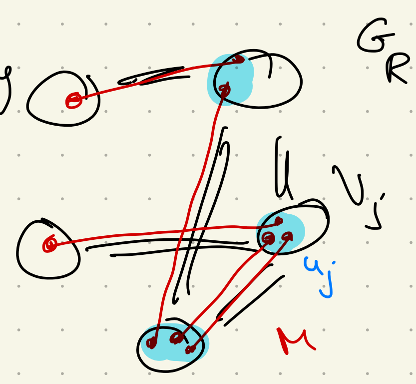
- Supp. not : - G union of n induced matchings
  - but $e(G) > cn^2$ for some $c > 0$.

- Apply reg. lem. on G w/ $\varepsilon = c/10$ and $m = 1/\varepsilon$
  Let $R = R(\varepsilon, 2\varepsilon)$ be a reduced gr.

- Do the standard cleaning $G \longrightarrow G_R \subseteq G$

$$e(G_R) \geq e(G) - 3\varepsilon n^2 \geq cn^2/2$$

- By Pigeonhole $\Rightarrow$ $\exists$ an induced matching M w/ $\geq cn/2$ edges in G_R
  Note that $|V(M)| \geq cn$



- Define $U_j = V_j \cap V(M)$ $\forall\, j \in [r] = V(R)$

Set $U = \cup \{ U_j : |U_j| \geq \varepsilon |V_j| \}$

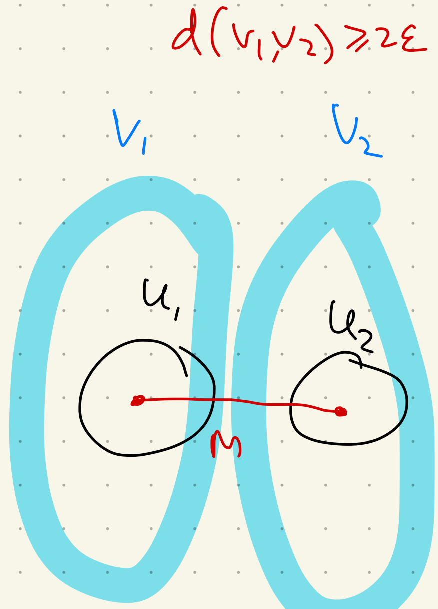
(collect the substantial parts)

From $\cup U_j = V(M)$ to U, we remove $\leq \varepsilon n = cn/10$ vts

$$\Rightarrow |U| > \frac{9}{10} |V(M)| = \frac{9}{5} |M|$$

So $|U| > |M|$ and $U \subseteq V(M)$

$\Rightarrow U$ spans an edge in M



- After the cleaning, we know this edge has to go btw some (U_1, U_2) in some ε-reg. pair (V_1, V_2) w/ density $\geq 2\varepsilon$.

- Since $|U_i| \geq \varepsilon |V_i|$, $i \in [2]$

ε-reg. $\Rightarrow d(U_1, U_2) \geq d(V_1, V_2) - \varepsilon \geq \varepsilon$

$\Rightarrow e(U_1, U_2) \geq \varepsilon |U_1||U_2| > |U_1|$

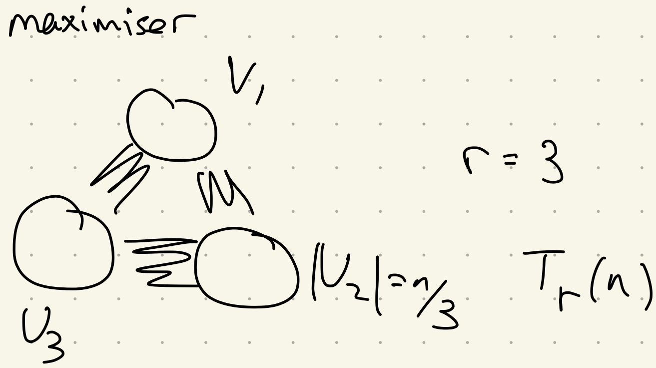
$\Rightarrow$ there is a non-M-edge in $[U_1, U_2]$ contradicting to M be induced. ☺

§ Ramsey-Turán for K_4 (Survey Ramsey-Turán theory Simonovits - Sós Disc. Math)

Motivation:

Turán: K_{r+1}-free $\Rightarrow$

The (unique) maximiser has a very rigid str.



In particular, $T_r(n)$ has $\Omega(n)$-size independent set.

Natural: What if no s.t. linear hole?

Initiated by Sós in 1969.

- Given H, $m, n \in \mathbb{N}$, the Ramsey-Turán number for H is

$$RT(n, H, m) = \max\left\{ e(G) : \begin{array}{l} \bullet\ |G| = n, \\ \bullet\ H\text{-free} \\ \bullet\ \alpha(G) \leq m \end{array} \right\}$$

Rmk: If there is no restriction, i.e. $m = n$, then $RT(n, H, n) = ex(n, H)$

The most classical case is when $m = o(n)$.

Def: Given a gr. H, let (Ramsey-Turán density of H)

$$\rho(H) := \lim_{\delta \to 0} \lim_{n \to \infty} \frac{RT(n, H, \delta n)}{\binom{n}{2}}$$

Define $RT(n, H, o(n)) = (\rho(H) + o(1)) \binom{n}{2}$.

Rmk By a simple ave. argument, the above limit exists.

Exer Prove that $RT(n, K_3, o(n)) = o(n^2)$

Compared to Mantel's thm. $ex(n, K_3) = \frac{n^2}{4}$.

Thm (Szemerédi 1973) $\rho(K_4) \le 1/4$

$$RT(n, K_4, o(n)) \le \frac{n^2}{8} + o(n^2).$$

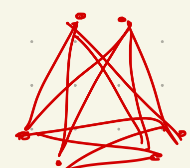
$ex(n, K_4) = \frac{n^2}{3} \pm O(1)$

Bollobás–Erdős graph, geometric const., matching lower bound

$$\Rightarrow \quad RT(n, K_4, o(n)) \geq n^2/8 - o(n^2)$$

We do not have an Erdős–Stone–Sim. for RT.

Open : Is $RT(n, K_{2,2,2}, o(n)) = o(n^2)$?



Sketch Pf Use Reg. lem to turn RT prob. into a weighted turán prob.

- G : - n-vx NTS
  - K_4-free $\Longrightarrow e(G) \le \frac{n^2}{8} + o(n^2)$
  - $\alpha(G) = o(n)$

$w(ij) = d(V_i, V_j)$

- Apply reg. lem on G and take its weighted reduced gr. R. $|R| = r$

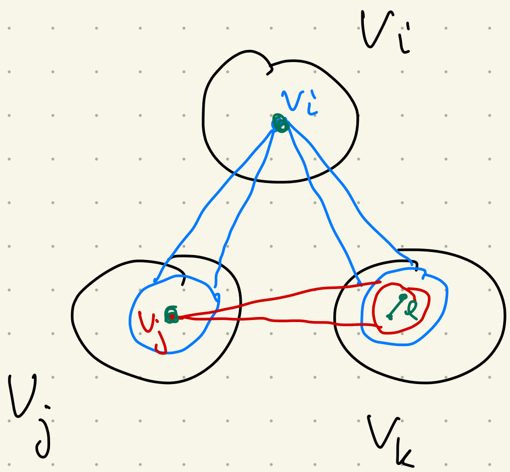
Suffices to show $\begin{cases} 1) & R : \Delta\text{-free} \\ 2) & \text{no edge in } R \text{ has weight larger than } 1/2 + o(1) \end{cases}$

as a gr.

1) $\Rightarrow$ # edges in $R \le r^2/4$.

2) $\Rightarrow e(G) \le \left(\frac{n}{r}\right)^2 \cdot \left(\frac{1}{2} + o(1)\right) \cdot r^2/4 = \left(\frac{1}{8} + o(1)\right) n^2$

1) if $\exists\ \Delta\ ijk$ in $R \rightsquigarrow V_i, V_j, U_k$



pairwise reg & <u>dense</u>

- typical $v_i \in V_i$

$$\Rightarrow \begin{matrix} |N(v_i) \cap V_j| \\ |N(v_i) \cap V_k| \end{matrix} = \Omega(n)$$

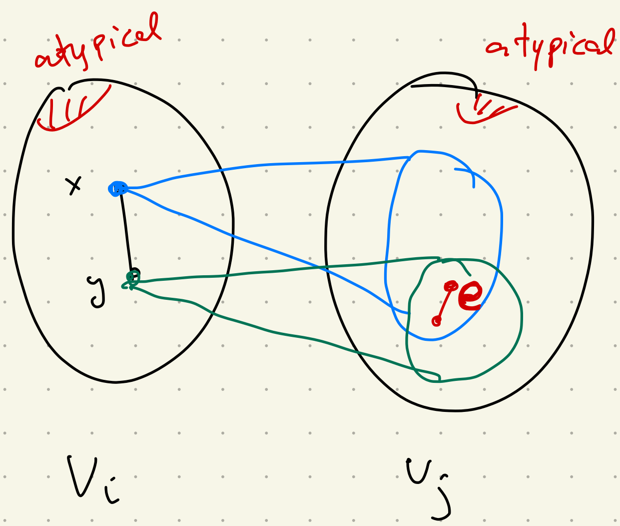
- pick typical v_j in $N(v_i) \cap V_j$.

$$\Rightarrow \quad |N(v_i; v_j) \cap V_k| = \Omega(n)$$

$\alpha(G) = o(n) \implies \exists$ edge $e \in N(v_i, v_j) \cap V_k$

e & $v_i, v_j \implies K_4$ ↯.

2) if $\exists\ w(ij) > \frac{1}{2} + c$, consider the corresponding (V_i, v_j)



- $\alpha(G) = o(n)$

$\Rightarrow \exists$ edge $xy \in V_1$

and x, y both typical (wrt V_2)

i.e. $d(x, V_j) \geq (\frac{1}{2} + c - \varepsilon)|V_j|$

$d(y, V_j) \geq (\frac{1}{2} + c - \varepsilon)|V_j|$

$d(V_i, V_j) > \frac{1}{2} + c$

$$\Rightarrow \underbrace{|N(x, y) \cap V_j|}_{\text{contains an edge } e \text{ as } \alpha(G) = o(n)} = \Omega(n)$$

$c \geq 2\varepsilon$

e & $xy \implies K_4$ ↯.

☺