Lecture 30

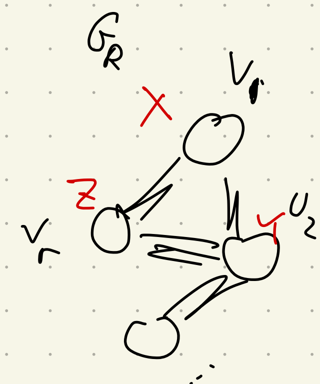
Pf (Δ removal lem). Suppose not true.

i.e. $\exists\, c>0$ s.t. $\forall\, a$, there is a counterexample G:

- G has $\leq an^3$ Δs
- the removal of any cn^2 edges does not make it Δ-free.

- Apply reg. lem. w/ $\varepsilon \ll c$ and $m = 1/\varepsilon$ to G to get an ε-reg. ptt. $V(G) = V_1 \cup \cdots \cup V_r$, where $m \leq r \leq M = M(\varepsilon, m)$, $\big||V_i| - |V_j|\big| \leq 1$ for all $i,j \in [r]$.

- Let $R = R(\varepsilon, c/4)$ be the reduced graph and $G_R \subseteq G$ the corresponding subgraph. Recall that $e(G) - e(G_R) \leq cn^2/2$



- By assumption, G_R contains a Δ, which lies in three distinct clusters, say X, Y, Z that are pairwise ε-reg. w/ density $\geq c/4$.

- By counting lem $\Rightarrow$ $G_R[X,Y,Z]$ contains at least

$$\left(\frac{n}{r}\right)^3 \left(\frac{c}{4} - O(\varepsilon)\right) \geq \left(\frac{c}{8r}\right)^3 n^3 \geq \left(\frac{c}{8M}\right)^3 n^3 \text{ triangles.}$$

Notice that $M = M(\varepsilon, m)$ depends in fact only on c.

Thus choosing $a = a(c) < \left(\frac{c}{8M}\right)^3 \Rightarrow G_R \subseteq G$ has $> an^3$ Δs $\lightning$.

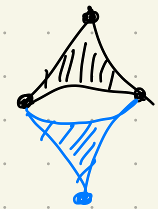
§ (6,3)-thm and Roth's thm.

- Ruzsa-Szemerédi (6,3)-thm $\Rightarrow$ Roth's thm on 3AP (3-term arithmetic progression)

- 3-uniform hypergraph $\mathcal{H} = (V, E)$, $E \subseteq \binom{V}{3}$

  is linear if any two of its edges share at most 1 vertex.

- For $s, t \in \mathbb{N}$, $\mathcal{H}$ contains an (s,t)-subgraph if $\exists$ s vertices in $\mathcal{H}$ inducing $\geq t$ edges.



(4,2)-subgraph

  $\mathcal{H}$ is (s,t)-free if it does not contain an (s,t)-subgraph.

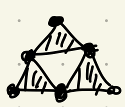
<u>Thm</u> (6,3)-thm: If an n-vertex 3-unif. hyp. $\mathcal{H}$ is (6,3)-free, $\Rightarrow$ then $e(\mathcal{H}) = o(n^2)$

<u>Rmk</u> 1) This upp bd. is not far from optimal:

$\exists$ n-vx 3-unif $\mathcal{H}$ that is (6,3)-free and $e(\mathcal{H}) \geq n^2 \cdot e^{-c\sqrt{\ln n}}$.

(larger than $n^{2-\varepsilon}$ $\forall$ const. $\varepsilon > 0$)

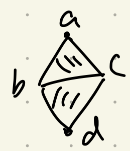
2) (6,3)-thm is equivalent to

$\mathrm{ex}(n, \{$   $\}) = o(n^2)$

<u>Pf</u>: Suppose not true: $\exists c > 0$ s.t. for infinitely many n, there is a (6,3)-free 3-unif. n-vx $\mathcal{H}$ w/ $e(\mathcal{H}) > cn^2$.

By zooming into a subgraph w/ higher average degree (which is still a counterexample), we may in addition assume that $\mathcal{H}$ is maximal in the sense

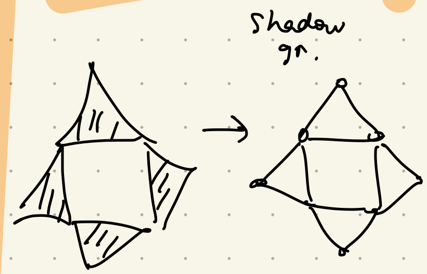
$$\max_{\mathcal{F} \subseteq \mathcal{H}} d(\mathcal{F}) = d(\mathcal{H})$$

- The maximality of $\mathcal{H} \Rightarrow \mathcal{H}$ is linear (i.e.  -free)
  - if $\exists$ 

    , then a,b,c,d form a component of $\mathcal{H}$

    b/c any other edge touching $\{a,b,c,d\}$ will yield a $(6,3)$-subg. $\lightning$.

    Now, $\mathcal{F} = \mathcal{H} - \{a,b,c,d\}$ has higher ave. deg. than $\mathcal{H}$ $\lightning$



- $\mathcal{H}$ Linear $\Longrightarrow e(\mathcal{H}) \leq \binom{n}{2}/3$
  (partial Steiner triple system)
- Let G be the shadow graph of $\mathcal{H}$
  - $V(G) = V(\mathcal{H})$
  - turn every hyperedge in $\mathcal{H}$ into a triangle in G.

A triangle in G is an $\mathcal{H}$-triangle if it corresponds to a hyperedge in $\mathcal{H}$.

- $\mathcal{H}$ is linear $\Longrightarrow$ $\mathcal{H}$-triangles in G are pairwise edge-disjoint

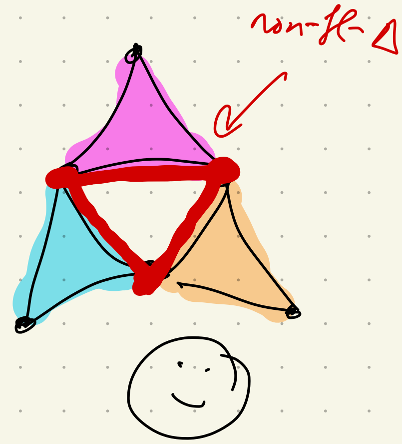
  $\Longrightarrow$ # pairwise edge-disj Δs in $G \geq e(\mathcal{H}) > cn^2$

$\overset{\text{removal lem}}{\Longrightarrow}$ G contains $\geq an^3$ Δs for some $a = a(c)$.

- For large n, $an^3 > n^2 > e(\mathcal{H})$

  $\Rightarrow \exists$ a non-$\mathcal{H}$-Δ in G.

  $\Rightarrow$ (6,3)-subgraph in $\mathcal{H}$ $\lightning$.



§ Roth's thm.

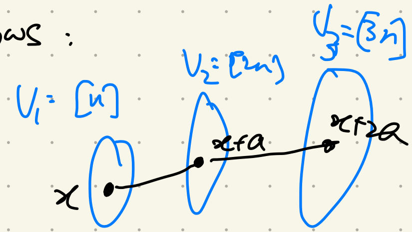
<u>Thm</u> (Roth's thm) $\forall \delta > 0$, $\exists n_0$ s.t. $\forall n \geq n_0$, any subset $S \subseteq [n]$ w/ size δn contains a 3AP.

$S \subseteq [n]$ 3AP-free $\Rightarrow$ $|S| = o(n)$

<u>Pf</u> ((6,3)-thm $\Rightarrow$ Roth's thm).

Suppose $\exists$ 3AP-free set $A \subseteq [n]$ w/ $|A| \geq \delta n$.

Def. a 3-partite 3-unif. hyp. $\mathcal{H}$ as follows:



- $V(\mathcal{H}) = \underset{V_1}{[n]} \cup \underset{V_2}{[2n]} \cup \underset{V_3}{[3n]}$
- $\forall a \in A$, $\forall x \in [n]$, add edge $(x, x+a, x+2a)$

<u>Observations</u>:
- $e(\mathcal{H}) = |A| \cdot n \geq \delta n^2$
- $\mathcal{H}$ is a linear hypergraph as every edge is completely determined by two pts $(x, x+a)$

- (6,3)-thm $\Rightarrow$ $\mathcal{H}$ has a (6,3)-subgraph.

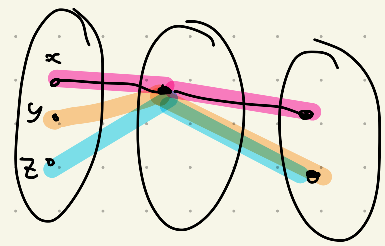
$(x,\ x+a,\ x+2a)$

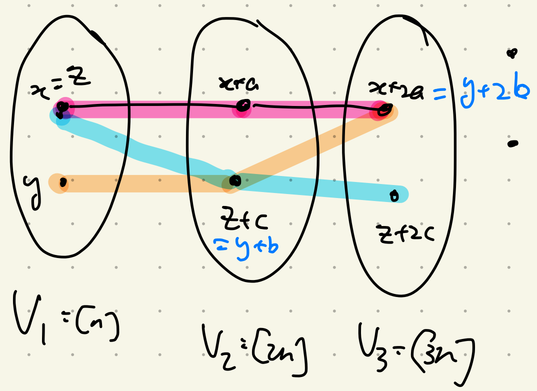
$(y,\ y+b,\ y+2b)$

$(z,\ z+c,\ z+2c)$

<u>Claim</u> $\mathcal{H}$ linear $\Rightarrow$ (6,3)-subgr. uses exactly two vxs from each V_i.

Pf:





- Wlog $x = z \neq y \Rightarrow x+a \neq z+c$, $x+2a \neq z+2c$
- wlog. $y+2b = x+2a$

$\Rightarrow y+b \neq x+a$

$y+b = z+c$

Calculation... $\Longrightarrow$ $b+c = 2a$, but $a, b, c \in A$

$\lightning$ A being 3AP-free.

Dense (6,3)-free 3-unif $\mathcal{H}$

In the above pf, A is 3AP-free $\Leftrightarrow$ $\mathcal{H}$ is (6,3)-free.

$e(\mathcal{H}) = |A| \cdot n$

- Behrend: $\exists$ 3AP-free $A \subseteq [n]$ of size $n \cdot e^{-c\sqrt{\log n}}$

$\Rightarrow \exists \mathcal{H}$ (6,3)-free w/ $e(\mathcal{H}) \geq n^2 \cdot e^{-c\sqrt{\log n}}$.

OPEN: Well-known conjecture. Simplest open case is $(7,4)$.

Conj (Brown - Erdős - Sós 73). Let $\in \mathbb{N}$

n-vx 3-unif $\mathcal{H}$: $(e+3,e)$-free $\Rightarrow$ $e(\mathcal{H}) = o(n^2)$

Conj (Long - Gowers) Let $\in \mathbb{N}$

$$(e+4,e)\text{-free} \Rightarrow e(\mathcal{H}) = O(n^{2-c}).$$

- Sárközy - Selkow : $(e+2+\lfloor \log_2 e \rfloor, e)$-free $\Rightarrow o(n^2)$
- Conlon - Gishboliner - Levanzov - Shapira

$$\left(e+O\left(\frac{\log e}{\log\log e}\right), e\right)\text{-free} \Rightarrow o(n^2).$$

- $(e+2,e)$-free : ref
  - [Delcourte - Postle]
  - [Glock - Joos - Kühn - Lichev - Pikhurko]
    $(6,4)$-problem
  - [Shangguan]