Lecture 29

- **Key lemmas**
  - embedding lem (embed any appropriate bdd deg graphs (up to linear-size))
  - counting lem (count accurately # fixed-size graphs)
  - removal lem

- Rmk: blowup lem allows embedding spanning bdd deg graph.

§1. Reduced graph

Def: (Reduced graph) Given an ε-reg partition $V(G) = V_0 \cup V_1 \cup \dots \cup V_r$ of G, $\delta > 0$, the reduced (cluster) graph $R = R(\varepsilon, \delta)$ of G is defined as follows

- $V(R) = [r]$
- $ij \in E(R) \iff (V_i, V_j)$ is ε-reg w/ density $\geq \delta$.

It is often helpful to view a reduced graph as a weighted graph, assigning weight $d_{ij} = d(V_i, V_j)$ to the edge ij in R.

We can also define the weighed deg. of a vx $i \in V(R)$ to be

$$d_R(i) = \sum_{j \in N_R(i)} d_{ij}.$$

Exer. Normalised min. deg is inherited by the reduced graph $R = R(\varepsilon, \delta)$, i.e.

$$\frac{\delta(R)+1}{r} \geq \frac{\delta(G)}{n} - \delta - \varepsilon \quad \left(= \frac{\delta(G)}{n} - o(1) \right)$$

Exer. Bound the edge-density of G by that of R's, i.e.

$$\frac{e(G)}{\binom{|G|}{2}} \leq \frac{e(R)}{\binom{|R|}{2}} + o(1)$$

The notion of reduced graph $R(\varepsilon,\delta)$ captures essentially the whole (asymp.) information of G on subgraph densities.

§2. Embedding lem.

Informally: if G is H-free $\Rightarrow$ R has no hom. image of H

i.e. $H \subseteq R(S) \Rightarrow H \subseteq G$.

Lem (Embedding lem) For any $d \in (0,1]$, $\Delta \geq 1$, $\exists\, \varepsilon_0 > 0$ s.t. T.F.H. for any $\varepsilon \leq \varepsilon_0$. Let G and H be graphs w/ $\Delta(H) \leq \Delta$, $s \in \mathbb{N}$ and $R = R(\varepsilon, d)$ be a reduced graph of G.

Suppose the corresponding reg. partition of G has each of its parts of size $\ell \geq 2s/d^{\Delta}$. Then if $H \subseteq R(S) \Rightarrow H \subseteq G$.

Rmk: Think of d, Δ as constants and $|G| = \Theta(\ell)$, $|H| = \Theta(s)$ and s could be linear in ℓ. That is we allow $|H| = \Omega(|G|)$

Sketch of pf. Choose $\varepsilon_0 < d$ s.t.

$$(d-\varepsilon_0)^{\Delta} - \varepsilon_0 \Delta \geq \frac{1}{2} d^{\Delta} \geq \varepsilon_0$$

Let $\varphi: V(H) \to V(R)$ be a hom. (exists as $H \subseteq R(S)$)

Order $V(H)$ as $u_1, u_2, \cdots, u_h$. Initially, set $Y_j = V_j$ for all $j \in [r]$.

Embed vxs $u_1, \dots, u_{i-1}$ one by one and update the set of eligible vxs $Y_{\varphi(u_j)} \subseteq V_{\varphi(u_j)}$ to embed to, for each u_j, $j \geq i$ and $u_{i-1}u_j \in E(H)$, by intersecting it w/ $N(u_{i-1})$, maintaining always $|Y_{\varphi(u_j)}| \geq \varepsilon |V_{\varphi(u_j)}|$.

When embedding u_i in $Y_{\varphi(u_i)} \subseteq V_{\varphi(u_i)}$, note that for each $j > i$ w/ $u_iu_j \in E(H)$, in $V_{\varphi(u_i)}$, all but $\varepsilon |V_{\varphi(u_i)}|$ vxs u satisfy

$$d(u, Y_{\varphi(u_j)}) \geqslant (d-\varepsilon)|Y_{\varphi(u_j)}| .$$

Since $|V_i| \cdot (d-\varepsilon)^{\Delta} - \varepsilon\Delta|V_i| \geqslant \max\{s, \varepsilon|V_i|\}$

we never get stuck.

__Exer__. Prove it rigorously for $H = \boxtimes = K_4^-$ and φ embeds H to a triangle in R.

__Exer__. Make the pf of the upper bd. of E-S-S. rigorous.

§3. Counting lem.

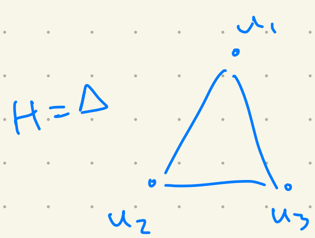
Lem (Counting lem) Given $H, V_1, \cdots, V_h$ w/ $h = |H|$ and $|V_i| = n$, all pairs (V_i, V_j) are ε-reg. and $d(V_i, V_j) = d_{ij} \gg \varepsilon$.

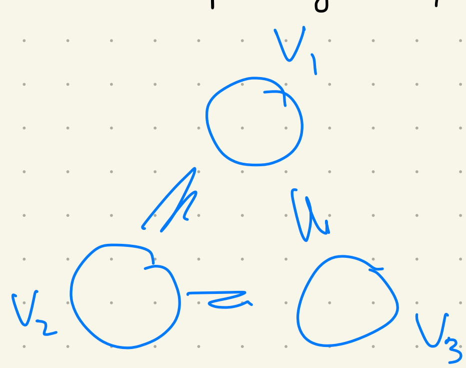
each V_i contains exactly one vx of H

Then # canonical copies of H in $V_1, \cdots, V_h$ is

$$\prod_{ij \in E(H)} (d_{ij} \pm C\varepsilon) \cdot n^h ,$$

where $C = C(H)$ is a const. depending only on H.





Exer. Prove the counting lem for the special case $H = K_3$.

§ 4. Ruzsa-Szemerédi triangle removal lem.

Informally: if a graph G is almost Δ-free ($o(n^3)$ triangle) then we can make G Δ-free by removing a negligible amount of edges ($o(n^2)$ edges).

Lem. (Rz-Sz Δ removal lem) $\forall c > 0$, $\exists a = a(c)$, s.t. for suff. large n T.F.H. Let G be an n-vx graph.
If G has $\leq an^3$ Δs, then G can be made Δ-free by removing $\leq cn^2$ edges.

Rmk: The contrapositive says one cannot destroy all Δs in a graph by removing few edges (i.e. there are many edge-disjoint triangles), then the graph has positive Δ-density.

More specifically, we have the following for all fixed H.

Removal lem (Contrapositive) $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. T.F.H. for all large n.
If G is an n-vx graph that is εn^2-far from being H-free (i.e. G contains $\geq \varepsilon n^2 / e(H)$ edge-disjoint copies of H)
then G contains $\geq \delta n^{|H|}$ copies of H.

Pf idea for removal lem:

By counting lem, if G is almost Δ-free, then its reduced graph R must be Δ-free. Then we can delete (few) edges not corresponding to R to make G Δ-free.

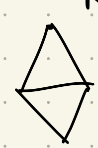
§ Cleaning the graph G.

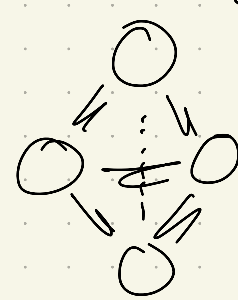
We shall define a subgraph $G_R \subseteq G$ corresponding to a reduced graph $R = R(\varepsilon, \delta)$, obtained by keeping only edges between (reg. and dense) pairs (V_i, V_j) for which $ij \in E(R)$.

Claim: $e(G_R) = e(G) - o(n^2)$

G

R





- Remove inner edges, i.e. edges in V_i, $i \in [r]$.

By choosing $m \geq 1/\varepsilon$ when applying the reg. lem.

$\Rightarrow$ # parts $= r \geq m \geq 1/\varepsilon$.

$\Rightarrow$ # inner edges $\leq r \cdot \binom{n/r}{2} \leq \frac{n^2}{2r} \leq \frac{n^2}{2m} \leq \frac{1}{2}\varepsilon n^2$.

- Remove edges btw. irreg. pairs.

irreg pairs $\leq \varepsilon r^2$

$\Rightarrow$ # edges btw irreg pairs $\leq \varepsilon r^2 \cdot \left(\frac{n}{r}\right)^2 = \varepsilon n^2$.

- Remove edges btw. sparse pairs (pairs not corresponding to edges in R)

$$\# \text{ such edges} \leq \delta \left(\frac{n}{r}\right)^2 \binom{r}{2} \leq \frac{1}{2}\delta n^2.$$

From $G \to G_R$, we delete at most

$$\frac{1}{2}(3\varepsilon + \delta)n^2 = O(\varepsilon + \delta)n^2 \text{ edges.}$$

Edges deleted in the above cleaning process is the non-essentially infor. we discard when forming the reduced graph.